Abstract:— A concept of a new set of features suitable for describing arbitrary shapes after their thinning is presented. The features are the minimal eigenvalues of the Toeplitz matrices formed from the rational functions, in which the coordinates of the characteristic pixels of the shape to be described are used as coefficients, the $x$ coordinates in the numerator polynomial, and the $y$ coordinates in the denominator polynomial. The rational function has a form which makes it a general measure of Image description and its features extraction. The method is introduced and applied for the first time.

Key Words:— Image Recognition, Image Shape Description, Computer Graphics, Feature Extraction, Toeplitz-Matrices Applications.

1 INTRODUCTION

The characteristics of Toeplitz matrices and their forms establish an interesting way of feature extraction and image description. This comes from the fact, that the determinants of Toeplitz matrices furnish a monotonically nonincreasing series of minimal eigenvalues [1,2]. The series converges to a limit of specific value depending on the characteristics of the input image, to which the matrices belong. The theory had been applied on recognition of scripts of cursive character, but with different definition of input data to the classifying system [3,4]. The approach showed similar satisfying results to those obtained by other methods of recognition [5,6]. The method presented in this paper, however, introduces a general way of describing images of both texts and pictures.

2 THEORETICAL APPROACHES

Consider the image drawn in Fig.1, whose thinned shape resembles the Arabic letter ى - yaa but without dots and handwritten ٔ، or a one-line drawn object like a jug, for example. For
the sake of classification and hence recognition, the image passes through some pre-processing
stages of filtering, thinning and, for some cases, segmentation.

![Image](image.png)

Fig. 1 A randomly drawn image.

After image preprocessing and extracting the image shape of Fig. 2a [7,8], the characteristic
points that show the main features defining the thinned image are assigned. The most feasible
considered points are usually those where exist cusps, edges, loops, windings, or any points of
a specific visible character (the points a to m in Fig. 2b are some examples).

![Image](image.png)

(a)             (b)

Fig. 2 The image of Fig. 1, thinned (a) and characterized for feature extraction (b).

2.1 The Feature Extraction Criterion

To extract the features from the image, the image is first binarized to define its feature points
as geometric ones. Each point is now having its x and y co-ordinates (Fig. 3). Therefore, the

![Image](image.png)

Fig. 3 The image of Fig. 2 thinned and projected on the x-y.
steps, in which the proposed criterion is found, are as follows:

1. Read the values of \( x_i \) and \( y_i \) co-ordinates of all \( n \) points \( (x_i, y_i) \), \( i=0,1,2,...,n \), which carry the specific features of the image.

2. Form the rational function

\[
H(z) = \frac{P(z)}{Q(z)} = \frac{x_0 + x_1 z + x_2 z^2 + ... + x_n z^n}{y_0 + y_1 z + y_2 z^2 + ... + y_n z^n}
\]  

in which the coefficients in the polynomials \( P(z) \) and \( Q(z) \) are the \( x \) and \( y \) co-ordinates of the points \( (x_i, y_i), i=0, 1, 2, ..., n \).

3. For the rational function of \( H(z) \) in Eq. (1) find Taylor series:

\[
T(z) = c_0 + c_1 z + c_2 z^2 + ... + c_n z^n + ...
\]  

The coefficients \( c_i, i = 0, 1, 2, ..., n \), are expressed by the coordinates of \( x_i \) and \( y_i \) of the points as follows:

\[
c_0 = \frac{x_0}{y_0}, \quad c_1 = \frac{1}{y_0^2} \begin{vmatrix} x_1 & y_1 \\ x_0 & y_0 \end{vmatrix},
\]

\[
c_2 = \frac{1}{y_0^3} \begin{vmatrix} x_2 & y_1 & y_2 \\ x_1 & y_0 & y_1 \\ x_0 & 0 & y_0 \end{vmatrix}, \quad c_3 = \frac{1}{y_0^4} \begin{vmatrix} x_3 & y_1 & y_2 & y_3 \\ x_2 & y_0 & y_1 & y_2 \\ x_1 & 0 & y_0 & y_1 \\ x_0 & 0 & 0 & y_0 \end{vmatrix}, \ldots
\]
The general form of $c_i$ for $i = 0, 1, 2, \ldots, n$ can therefore be given by the following form:

$$
c_i = \frac{1}{y_0^{i+1}} \begin{vmatrix}
    x_i & y_1 & y_2 & \cdots & y_i \\
    x_{i-1} & y_0 & y_1 & \cdots & y_{i-1} \\
    x_{i-2} & 0 & y_0 & \cdots & y_{i-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    x_0 & 0 & 0 & \cdots & y_0
\end{vmatrix}.
$$

(3)

4. Determine Toeplitz forms. The determinants of their matrices are evaluated below.

$$
D_0 = c_0 = \frac{x_0}{y_0}, \quad D_1 = \begin{vmatrix}
    c_0 & c_1 \\
    c_1 & c_0
\end{vmatrix}, \quad D_2 = \begin{vmatrix}
    c_0 & c_1 & c_2 \\
    c_1 & c_0 & c_1 \\
    c_2 & c_1 & c_0
\end{vmatrix}, \ldots, \quad D_n = \begin{vmatrix}
    c_0 & c_1 & c_2 & \cdots & c_n \\
    c_1 & c_0 & c_1 & \cdots & c_{n-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_n & c_{n-1} & c_{n-2} & \cdots & c_0
\end{vmatrix}.
$$

And in general, the $i^{th}$ determinant is given by:
5. Once the determinants have been calculated, determine their eigenvalues \([9]\) and evaluate the lowest (minimal) eigenvalue for each determinant. For the \(i^{th}\) determinant \(D_i\), the lowest eigenvalue is \(\lambda_{\min_1}\), designated simply by \(\lambda_i\):

\[
\lambda_{\min_1} \{D_i\} = \lambda_{\min_1} = \lambda_i, \quad i = 0, ..., n. \tag{5}
\]

\(\lambda_i\) for \(i = 0, 1, 2, ..., n\) form a monotonically nonincreasing series \([2]\), i.e.,

\[
\lambda_0 \geq ... \geq \lambda_i \geq ... \geq \lambda_n, \quad i = 1, ..., n-1
\]

Experiments \([2,10]\) have shown that the difference between the successive values \(\lambda_i\) decreases rapidly and for a low number of components \(n\) in Eq.(4), usually not more than 3, the difference value tends to a reference zero, \(\epsilon\), say.

\[
|\lambda_i - \lambda_{i+1}| \leq \epsilon, \tag{6}
\]

The reference zero \(\epsilon\) in Eq.(6) is specifically chosen for each set of the minimal eigenvalues of the reference image. In most cases, \(\epsilon\) has no larger value than \(10^{-4}\), which very often takes place for \(n\) having a value not exceeding 15.

2.2 Image-Vector Matching for Similarity and Recognition

Unknown images are classified for the sake of recognition by comparing the feature vector of the unknown image with that of the reference one. Here we consider Absolute Deviation method of approximation and comparison with two numerical examples to support the theory introduced in this paper.
Similarity Measure Techniques

The eigenvalues $\lambda_i$ in Eq.(5) form a very specific set of numbers; they are the elements of the feature vector, which describes some specific characteristics of a certain image. Depending on the values of the elements of the unknown vector and their relation to the reference image vector, the evaluation of similarity (or dissimilarity) and classification takes place.

Definition 1 - Similar Images

Two images are said to be similar if the minimum eigenvalues in Eq.(5) of their corresponding Toeplitz Matrices are of the same values.

Therefore, if $\Psi$ and $\Phi$ are the feature vectors of two images, then the images are said to be similar if they satisfy the following condition

$$|\lambda_i^\Psi - \lambda_i^\Phi| = 0 \text{ for } i = 1, 2, 3, ..., n$$

(7)

Accordingly, the following experimentally demonstrated assertion holds:

Assertion 1

Let $\Phi$ be the feature vector of the reference image (template in the database). Its elements are the minimal eigenvalues $\lambda_i, i = 1, ..., n$ with $\lambda_n^\Phi$ to be its limit, i.e.,

$$\Phi = [\lambda_1^\Phi, \lambda_2^\Phi, ..., \lambda_n^\Phi]$$

And let $\Psi$ be the feature vector of the unknown image whose elements are the minimal eigenvalues $\lambda_j, j = 1, ..., n$ having the limit $\lambda_n^\Psi$, i.e.,

$$\Psi = [\lambda_1^\Psi, \lambda_2^\Psi, ..., \lambda_n^\Psi]$$

Then

$$|\lambda_i^\Psi - \lambda_i^\Phi| \leq \varepsilon \text{ for } i = 1, 2, 3, ..., N$$

(8)

is a necessary and sufficient condition for the feature vectors $\Psi$ and $\Phi$ to be similar, where $N$ is the value at which the condition of Eq.(6) is satisfied.
Remark 1
Although in [11] the value of N was verified and the experiments showed that a selection of \( N=15 \) was sufficient, Eq.(7) and hence Eq.(8) are, theoretically, satisfied at \( N=n \) and \( n \to \infty \). This can be discussed as follows [1,11]:

Recall that if Eq.(1) is an analytic regular function with a greatest lower bound of \( R(f) \), then

\[
\lim_{n \to \infty} \lambda_n = R(f).
\]

Therefore, from theoretical point of view, Eq.(8) is formulated as follows:

\[
|\hat{\lambda}_i,_{\Psi} - \hat{\lambda}_i,_{\Phi}| = 0 \quad \text{for all values of } i, \ i = 1, 2, 3, ..., N
\]

with \( N = n \), and \( n \to \infty \).

2.3 Classification Analysis
For comparison and classification of images for the aim of their recognition, some known approximation methods were used. Here introduced the Absolute Deviation method with its application to the author's criterion in testing for minimal difference between the unknown image and the reference one.

1) Absolute Deviation [12,13] - Given two patterns \( \Psi \) and \( \Phi \), defined as in Assertion 1, with their elements to furnish the graphs shown in Fig.4. The summation of the absolute-values of the differences between \( \Psi \) and \( \Phi \) elements for \( i = 1 - N \) is called the Absolute-Deviation and given by Eq.(9).

\[
g(a,b) = \sum_{i=1}^{N} |\Psi - \Phi| \quad (9)
\]

with \( \Psi = y_i \) and \( \Phi = ax_i + b \), the approximating line.

Fig.4 \( \Psi \) and \( \Phi \) in one plane for linear approximation.
Absolute Deviation approximation requires that \( g \) in Eq.(9) is minimum. Multivariate calculus [12] implies that, for \( g \) to be minimum at \((a,b)\) and hence, \( \Psi \) and \( \Phi \) are similar it is necessary that the following two conditions are satisfied:

\[
\frac{\partial}{\partial a} g(a,b) = 0 \quad \text{and} \quad \frac{\partial}{\partial b} g(a,b) = 0
\]  

(10)

Consequently, if Eqs.(10) has the solution \( a = A \) and \( b = B \), then

\[
y = Ax + B
\]  

(11)

is the characteristic equation for the minimum of \( g \).

3  EXAMPLES, APPLICATIONS AND COMPUTER IMPLEMENTATION OF THE THEORY

**Example 1 - Image comparison and recognition**

Given three images (Fig.5) thinned by the algorithms in [7,8]. It is required to classify and recognise each of them. All of them have similar shapes to that of the image in Fig.1 after skeletonization (Fig.2a). To follow the theory procedure suggested in this paper for main feature-extraction and similarity measurement, apply Assertion 1. Then search for the minimal difference of Eq.(9) to find the characteristic equation given by Eq.(11) for each image. The characteristic equation is to be compared with the reference one in the data-base. Notice that in

Fig.5  Three different shapes assumed to have similar features; they resemble:

(a) The Arabic handwritten letter \( \text{ل} - \text{yaa}, \)
(b) a jug.  (c) a pot.

Fig.5 exist two different classes; (a) shows an image which belongs to Arabic-script class, while (b) and (c) show images of drawn-picture class. The feature vectors of the three images are given in Fig.6 through their minimal eigenvalues \( (\lambda_i) \). Notice that although the jug has many similar features to those of the pot, there still can be seen some difference between them.
Hence, each of the shapes is recognised as it should be. Moreover, the script *yaa*, which actually belongs to another different class, shows extremely different feature-vector from the two other images.

**Fig.6** The minimal eigenvalues of the shapes in Fig.5 furnish dissimilarities between different images.

**Example 2 - Application to script recognition**

Consider the printed letter *B* in five different bold fonts, each with its machine cursive form. These letters and their thinned shapes are given in Fig.7. For all the ten fonts the feature vectors are determined.

**Fig.7** The letter *B* in five different fonts B₁ - B₅ with their italics style, bitmapped (a) and thinned (b).
The computed results are given in Fig.8, on one plane for comparison. All are of similar behaviour. However, some are so similar that they have identical graphs (2b and 4b).

Notice that for the aim of recognition when comparing with reference patterns of the script B we get the feature vectors of Fig.9. They all are similar and belong to the same standard reference image.
Computer Program
The algorithm has been programmed. Its essential and most significant steps begin by following and solving Eqs. (2) to (5) to determine the values of minimal eigenvalues, the elements of the feature vector required for classification. Then, to classify the shapes of Fig. 5, the program proceeds with Eqs. (9) to (11). Examples 1 and 2 are direct applications of the program, whose results are those shown in Fig. 9.

4 CONCLUSIONS AND FUTURE WORK

The results of computing showed high efficiency of the classifying algorithm. The differences between the feature vectors of the different considered shapes give a simple practical proof of the theory. The basics of the algorithm were successfully used in different approaches for recognition like postcode, signature and many other applications [4]. However, the new modification given in this paper is supposed to make it easier to input the data to the system of classification and recognition from a large variety of patterns. However, it is of great importance to have a system not only general and fast, but invariant to some shape-change like rotation, scaling or number and place of characteristic points, as well. Therefore, to extend the applications of the method, the author has been considering its invariance to those and other transformations. On the whole, the algorithm is invariant to parallel shifting and coordinate-scaling, but still sensitive to some other changes in its current form; in some complex cases, for higher efficiency, the determinants had to have much larger dimensions to get more feature vectors which in turn increased the time and hence cost of operations. In conclusion, despite the so far satisfying results, the implemented algorithm still needs more improvement and modification to match and meet both theoretical and practical requirements. In fact, this is the subject of the author's current research and very near future publications.

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References:


