THE PRIME RADICAL OF CONSTANTS OF ALGEBRAIC $q$-SKEW DERIVATIONS

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Abstract. Let $\delta$ denote a $q$-skew $\sigma$-derivation of an algebra $R$ and $R^{(\delta)} = \{ r \in R \mid \delta(r) = 0 \}$ stand for the subalgebra of invariants. We prove that the $\sigma$-prime radical of $R^{(\delta)}$ is nilpotent provided $R$ is $(\sigma, \delta)$-semiprime and the action of $\delta$ on $R$ is algebraic.

It is known that if a semiprime algebra $R$ is acted by an algebraic derivation $\delta$ then the subalgebra of constants $R^{(\delta)}$ does not have to be semiprime. In [5], the first named author proved that in the above situation the prime radical of $R^{(\delta)}$ is nilpotent with the index of nilpotency depending on the minimal polynomial of $\delta$. It was also proved in [5] that the analogous result holds for the action of algebraic automorphisms under the extra assumption that the characteristic of $R$ is equal to 0.

The recent interest in the actions and invariants of Hopf algebras caused extensive investigations of properties of constants under the action of $q$-skew derivations. We show in Theorem 9 that if an algebraic $q$-skew $\sigma$-derivation $\delta$ acts on a $(\sigma, \delta)$-semiprime algebra $R$, then the $\sigma$-prime radical of the subalgebra of constants $R^{(\delta)}$ is nilpotent. Thus, in particular, if both $\delta$ and $\sigma$ are algebraic, the prime radical of $R^{(\delta)}$ is nilpotent.

Notice that $\text{id}_R - \sigma$ is $1$-skew $\sigma$-derivation of $R$ for any automorphism $\sigma$ of $R$. Hence Theorem 9 applies also to the action of algebraic automorphisms and no assumptions on the characteristic of $R$ are necessary. Also when $\sigma = \text{id}_R$, i.e. $\delta$ is a usual derivation, our result generalizes the result form [5] mentioned above to the case when $R$ is a $\delta$-semiprime algebra.

1. PRELIMINARIES

Let $R$ be an associative algebra over a field $K$ and let $\sigma$ be a $K$-linear automorphism of $R$. Recall that a $K$-linear map $\delta: R \to R$ is a $\sigma$-derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s),$$
for all \( r, s \in R \). Furthermore, we say that \( \delta \) is a \( q \)-skew \( \sigma \)-derivation if there exists a nonzero \( q \in K \) such that \( \delta \sigma = q \sigma \delta \). When we refer to \( \delta \) or \( \sigma \) as being algebraic, we mean that they are algebraic as \( K \)-linear transformations of \( R \). Subsets \( A \) of \( R \) such that \( \sigma(A) = A \) and \( \delta(A) \subseteq A \) are known as \( \sigma \)-stable and \( \delta \)-stable, respectively. Subsets satisfying both properties are called \((\sigma, \delta)\)-stable. If \( A \) is a \((\sigma, \delta)\)-stable subset of \( R \), we let

\[
A^{(\delta)} = \{ a \in A \mid \delta(a) = 0 \}
\]
denote the invariants of \( A \).

Throughout the paper \( R \) will always denote a \((\sigma, \delta)\)-semiprime algebra over a field \( K \), i.e. we will assume that \( R \) does not contain nonzero nilpotent \((\sigma, \delta)\)-stable ideals. In fact (cf. [6]) this is equivalent to assuming that \( R \) does not have nonzero nilpotent ideals \( I \) such that \( \sigma(I) \subseteq I \) and \( \delta(I) \subseteq I \). \( \delta \) always stand for an algebraic \( q \)-skew \( \sigma \)-derivation of \( R \). For subsets \( A, B \) of \( R \), \( \text{lann}_A(B) \) denotes the left annihilator of \( B \) in \( A \), i.e. \( \text{lann}_A(B) = \{ a \in A \mid aB = 0 \} \). Similarly \( \text{rann}_A(B) = \{ a \in A \mid Ba = 0 \} \). We say that a \((\sigma, \delta)\)-stable ideal \( J \) of \( R \) is \((\sigma, \delta)\)-essential if \( J \) has nonzero intersection with every nonzero \((\sigma, \delta)\)-stable ideal of \( R \). One can easily check using \((\sigma, \delta)\)-primeness of \( R \), that a \((\sigma, \delta)\)-stable ideal \( J \) of \( R \) is \((\sigma, \delta)\)-essential if and only if \( \text{lann}_R(J) = 0 \) and \( I + \text{lann}_R(I) \) is \((\sigma, \delta)\)-essential with \( I \cap \text{lann}_R(I) = 0 \) for any \((\sigma, \delta)\)-stable ideal \( I \).

In the sequel we will use the following well-known (cf. [3]) property of binomial symbols:

**Lemma 1.** Let \( p \) be a prime and \( k, l \) nonnegative integers such that \( k = \sum_{i=0}^{n} k_i p^i \) and \( l = \sum_{i=0}^{n} l_i p^i \), where \( 0 \leq k_i, l_i < p \). Then

\[
\binom{k}{l} \equiv \prod_{i=0}^{n} \binom{k_i}{l_i} \mod p.
\]

In particular \( \binom{kp^s}{lp^s} \equiv \binom{k}{l} \mod p \) for every \( s \in \mathbb{N} \) and \( \binom{k}{l} \equiv 0 \mod p \) provided \( p \) divides \( k \) and is relatively prime to \( l \).

Recall that for a nonzero \( q \in K \) and integers \( k, l \) one can define \( q \)-binomial symbols \( \binom{k}{l}_q \) as follows:

\[
\binom{k}{l}_q = 0 \text{ unless } 0 \leq l \leq k; \quad \binom{0}{l}_q = 1 \quad \text{and for } 0 \leq l \leq k \text{ the symbol } \binom{k}{l}_q \text{ is defined inductively by } \binom{k}{l}_q = q^l \binom{k-1}{l-1}_q + \binom{k-1}{l-1}_q.
\]

Moreover, it is known that \( \binom{k}{l}_q \), for \( 0 \leq l \leq k \), is equal to the evaluation at \( x = q \) of the polynomial function \( \frac{(x^k-1)(x^{k-1}-1)\ldots(x^{k-l+1}-1)}{(x-1)(x^{l-1}-1)\ldots(x^{k-l+1}-1)} \). In the following lemma we collect the properties of \( q \)-binomial symbols which will be used later.

**Lemma 2.** ([7], Proposition 1, Corollary 2) Let \( 0 \leq l \leq k \). Then:

1. Suppose that \( q \) is a primitive \( N \)-th root of unity. Then

\[
\binom{k}{l}_q = \binom{k_R}{l_R}_q \binom{k_D}{l_D}_q
\]

where the integers \( n_D, n_R \) for \( n = k, l \) are uniquely determined by \( n = n_DN + n_R \), where \( 0 \leq n_R < N \).

2. Suppose that \( q \in K \setminus \{0\} \) and \( k > 1 \). Then \( \binom{k}{l}_q = 0 \) for all \( 1 \leq l \leq k \) if and only if one of the following conditions holds:

(i) \( \text{char}K = 0 \) and \( q \) is a primitive \( k \)-th root of unity.
(ii) \( \text{char} K = p > 0 \) and \( k = Np^s \), where \( N \geq 1 \), \( s \geq 0 \) and \( q \) is a primitive \( N \)-th root of unity.

2. THE ALGEBRA \( R\{x\} \)

Throughout this section we additionally assume that the \( \delta \) is a nilpotent \( q \)-skew \( \sigma \)-derivation of \( R \). \( R[X; \sigma, \delta] \) denotes the skew polynomial ring with coefficients written on the left. It is known (cf. [4]) that \( \sigma \) has the unique extension to an automorphism of \( R[X; \sigma, \delta] \) given by \( \sigma(X) = q^{-1}X \). Moreover, it is clear that \( \delta \) extends to an inner \( q \)-skew \( \sigma \)-derivation of \( R[X; \sigma, \delta] \) adjoint to \( X \).

Let us consider the left \( R[X; \sigma, \delta] \) module \( M = R[X; \sigma, \delta]/R[X; \sigma, \delta]X \). Then \( X \) acts as \( \delta \) on \( M \). Since \( \delta \) is nilpotent, the annihilator \( A \) of \( M \) is an ideal of \( R[X; \sigma, \delta] \) which contains a nonzero power of \( X \), has zero intersection with the coefficient algebra \( R \) and is \( (\sigma, \delta) \)-stable.

Let \( B \) be a maximal \( (\sigma, \delta) \)-stable ideal of \( R[X; \sigma, \delta] \) in the class of ideals containing \( A \) and having zero intersection with \( R \). Let \( R\{x\} \) denote the factor algebra \( R[X; \sigma, \delta]/B \) and let \( x \) be the image of \( X \) in \( R\{x\} \). It is clear, by the construction, that \( R \) embeds into \( R\{x\} \) and that \( R\{x\} \) is \( (\sigma, \delta) \)-semiprime when \( R \) is such. Note also that \( \sigma \) can be extended to an automorphism of \( R\{x\} \) such that \( \sigma(x) = q^{-1}x \) and \( \delta \) can be viewed as an inner \( q \)-skew \( \sigma \)-derivation of \( R\{x\} \) induced by a nilpotent element \( x \). We let \( m \) to denote the index of nilpotency of \( x \).

We begin with the following simple observation:

**Lemma 3.** Let \( J \) be a \( (\sigma, \delta) \)-stable ideal of \( R \). Then for any \( k \in \mathbb{N} \) we have:

1. \( \text{l.ann}_J(x^k) \) is \( (\sigma, \delta) \)-stable.
2. \( \text{l.ann}_J(x^k) = 0 \) if and only if \( \text{r.ann}_J(x^k) = 0 \)

**Proof:** (1). Since \( \sigma(x) = q^{-1}x \), \( A = \text{l.ann}_J(x^k) \) is clearly \( \sigma \)-stable. Moreover for \( r \in A \) we have \( \delta(r)x^k = (xr - \sigma(r)x)x^k = 0 \). This gives (1).

(2). The situation is left-right symmetric, so it is enough to prove one implication from (2). Suppose \( A = \text{l.ann}_J(x^k) \neq 0 \). Since \( A \) is \( (\sigma, \delta) \)-stable and \( \delta \) is nilpotent, \( A(\delta) \neq 0 \). Then \( 0 \neq A = \sigma^{-1}(A) \subseteq \text{r.ann}_J(x^k) \) and (2) follows. \( \square \)

We will frequently use the following rules (cf. [4, 2.5] and [2, Lemma 5.1]):

\[
(1) \quad x^kr = \sum_{i=0}^{k} \binom{k}{i}_q \sigma^{k-i} \delta^i(r)x^{k-i}
\]

and

\[
(2) \quad \delta^k(r) = \sum_{i=0}^{k} (-1)^i q^{m_i} \binom{k}{i}_q x^{k-i} \sigma^i(r)x^i,
\]

for any \( r \in R \) and any nonnegative integer \( k \), where \( m_i \)'s are suitably chosen integers.

Since \( x \) is nilpotent, for any nonzero \( (\sigma, \delta) \)-stable ideal \( J \) of \( R \) there exists \( k \in \mathbb{N} \) such that \( \text{l.ann}_J(x^k) \neq 0 \). The following lemma shows that such minimal \( k \) is of very special form.

**Lemma 4.** Let \( J \) be a nonzero \((\sigma, \delta)\)-stable ideal of \( R \) and \( k \in \mathbb{N} \) be the smallest number such that \( \text{l.ann}_J(x^k) \neq 0 \). Then either \( k = 1 \) or \( k > 1 \) and \( \binom{k}{i}_q = 0 \) for all \( i \in \{1, 2, \ldots, k - 1\} \).

**Proof:** Suppose that \( k > 1 \) and there exists \( 1 \leq j \leq k - 1 \) such that \( \binom{k}{j}_q \neq 0 \). Let \( j \) be the smallest such number. Set \( A = \text{l.ann}_J(x^k) \). Then, by Lemma 3(1), \( A \) is \((\sigma, \delta)\)-stable. Because
A annihilates \(x^k\), the formula (1) yields that \(0 = Ax^k Ax^{k-1} = A\delta^k(A)x^{k-1}\). Hence, by the choice of \(k\), \(A\delta^k(A) = 0\).

Multiplying the formula (2) on the right by \(x^{k-j-1}\) we obtain
\[
delta^k(r)x^{k-j-1} = x^{k-j}x^{k-j-1} + (-1)^j q^{m_j} \left( \binom{k}{j} q \right) x^{k-j}\sigma^j(r)x^{k-1}
\]
for all \(r \in J\). Hence, by the above, \(0 = A\delta^k(A)x^{k-j-1} = Ax^{k-j}Ax^{k-1}\). Therefore, since \(A\) is \((\sigma, \delta)\)-stable, we also have \(Ax^{k-1}Ax^{k-1} = 0\). However notice that \(Ax^{k-1}\) is a \((\sigma, \delta)\)-stable left ideal of \(R\{x\}\). Since \(R\{x\}\) is \((\sigma, \delta)\)-semiprime, \(Ax^{k-1} = 0\). This contradicts the assumption and completes the proof of the lemma.

\[\square\]

The above lemma together with Lemma 2(2) give us the following:

**Corollary 5.** Let \(J\) be a nonzero \((\sigma, \delta)\)-stable ideal of \(R\) and \(k \in \mathbb{N}\) be the smallest number such that \(\lann J(x^k) \neq 0\). Then:

1. If \(q\) is not a root of unity then \(k = 1\).
2. If \(q\) is a primitive \(N\)-th root of unity then:
   - (i) if \(\text{char} K = 0\) then either \(k = 1\) or \(k = N\).
   - (ii) if \(\text{char} K = p > 0\) then either \(k = 1\) or \(k = Np^s\) for some \(s \geq 0\).

We finish this section with one more technical result.

**Lemma 6.** Let \(w \in \mathbb{N}\) and \(T: R \to R\{x\}\) be given by the formula
\[
T(r) = \sum_{l=0}^{w} q^{-w l} x^{w-l} \sigma^l(r) x^l.
\]
Suppose that \(q\) is a primitive \(N\)-th root of unity. Then:

1. If \(\text{char} K = 0\), \(N \geq 2\) and \(w = tN - 1\) for some \(t \geq 1\), then \(T\) can be written in the form
   \[
   T(r) = \sum_{l=1}^{t} \binom{l}{t} \sigma^{(t-1)N} \delta^{N-1}(r) x^{(t-1)N}.
   \]
2. If \(\text{char} K = p > 0\), \(N \geq 1\) and \(w = tNp^s - 1\) for some \(t \geq 1\) and \(s \geq 0\), then \(T\) can be written in the form
   \[
   T(r) = \sum_{l=1}^{t} \binom{l}{t} \sigma^{(t-1)Np^s} \delta^{Np^s-1}(r) x^{(t-1)Np^s}.
   \]

**Proof:** By making use of the formula (1) we can rewrite the formula for \(T(r)\) in the form
\[
T(r) = \sum_{l=0}^{w} S_l \sigma^{w-l} \delta^l(r)x^{w-l}, \text{ where } S_l = \sum_{k=l}^{w} \binom{k}{l} q^{-(w-k)(w-l)}.
\]
Moreover, one can check that
\[
(1 - q^{w-l})S_l = q^{2(w-l)+1} S_{l-1} - q^{w-l} \binom{w+1}{l}.
\]
(i) Suppose that \( \text{char} K = 0, \ N \geq 2 \) and \( w = tN - 1 \) for some \( t \geq 1 \). In order to prove (i) it is enough to show that

\[
S_j = 0 \quad \text{for} \quad j \notin \{N - 1, 2N - 1, \ldots, tN - 1\} \quad \text{and} \quad S_{IN^t - 1} = \binom{t}{l} \quad \text{for} \quad 1 \leq l \leq t.
\]

(5)

Notice that \( S_{tN-1} = \binom{tN-1}{tN-1} = 1 \) and \( S_{tN-2} = q^{-1} + \binom{tN-1}{tN-2} = q^{-1} + 1 + q + \ldots + q^{tN-2} = 0. \)

Suppose that \( S_j = 0 \) and \( j - 1 \notin \{N - 1, 2N - 1, \ldots, tN - 1\} \). Then \( j = jDN + jR \) for uniquely determined integers \( jD, jR \), where \( 0 < jR < N \). Then, by Lemma 2(1), we have \( \binom{tN}{j} = \binom{tN}{jR} \binom{tN}{jD} = 0. \) Hence, by (4) applied to \( l = j \) and Lemma 2(1) we get \( S_{j-1} = S_{kN-1} = q^{(tN)(\binom{tN}{kN})_q} = \binom{t}{l} \).

Finally suppose that \( j = kN - 1 \) for some \( 1 \leq k \leq t \) and \( S_j = S_{kN-1} = \binom{t}{l} \). Then applying (4) for \( l = j \) we obtain \( 0 = qS_{j-1} - \binom{tN}{kN} \). Notice that, by Lemma 2(1), \( \binom{tN}{kN} = \binom{0}{N-1} \binom{0}{k} = 0 \) as, by the assumption, \( N \geq 2 \). Thus \( S_{j-1} = 0 \). This completes the proof of (5).

(ii) Suppose \( \text{char} K = p > 0, \ N \geq 1 \) and \( w = tNp^s - 1 \) for some \( t \geq 1 \) and \( s \geq 0 \). In order to prove (ii) it is enough to show that

\[
S_j = 0 \quad \text{for} \quad j \notin \{Np^s - 1, 2Np^s - 1, \ldots, tNp^s - 1\} \quad \text{and} \quad S_{INp^s - 1} = \binom{t}{l} \quad \text{for} \quad 1 \leq l \leq t.
\]

(6)

If \( s = 0 \) and \( N = 1 \), then \( w = t - 1 \) and the formula (4) reduces to \( S_{t-1} = \binom{t}{l} \) for \( 1 \leq l \leq t \). This gives (6) in this case.

If \( s = 0 \) and \( N \geq 2 \), then the proof of (6) is the same as that of (5). Thus we may assume that \( s \geq 1 \). One can also check that \( S_{tNp^s-1} = 1 \) and \( S_{tNp^s-2} = 0. \)

Suppose that \( S_j = 0 \) and \( j - 1 \notin \{Np^s - 1, 2Np^s - 1, \ldots, tNp^s - 1\} \). Let \( j = jDN + jR \) with \( 0 \leq jR < N \). If \( jR \neq 0 \) then \( \binom{tN}{jR} = 0. \) If \( jR = 0 \) then \( p \) does not divide \( jDN \) as \( N \) and \( p \) are relatively prime. Then Lemma 1 and the assumption on \( K \) yield \( \binom{tN}{jD} = 0 \). Thus, in any case, \( \binom{tNp^s}{j} = \binom{tNp^s}{jD} = 0 \). Hence, by (4) applied to \( l = j \), we get \( S_{j-1} = 0 \).

Suppose \( S_j = 0 \) and \( j - 1 = kNp^s - 1 \), where \( 1 \leq kNp^s - 1 \). Then, as in the proof of (5), we obtain \( S_{j-1} = S_{kNp^s-1} = q^{(tNp^s)(\binom{tNp^s}{kNp^s}_q) = \binom{tNp^s}{kNp^s}} = \binom{t}{l} \), where the last equality is given by Lemma 1.

Finally suppose that \( j = kNp^s - 1 \) for some \( 1 \leq k \leq t \) and \( S_j = S_{kNp^s-1} = \binom{t}{l} \). By Lemma 2(1), \( v = \binom{tNp^s}{kNp^s-1}_q = \binom{tNp^s}{kNp^s-1} \). Notice that \( v = 0. \) Indeed: if \( N \geq 2 \) then \( \binom{tNp^s}{kNp^s-1}_q = 0 \) and for \( N = 1 \), since \( s \geq 1 \), \( v = \binom{tNp^s}{kNp^s-1} = 0 \) by Lemma 1. Now \( S_{j-1} = 0 \) follows as in the proof of (5). This completes the proof of the lemma.
3. THE PRIME RADICAL OF $R^{(δ)}$

With the technical preparation made in the previous section we are ready to deal with the prime radical of the algebra of constants $R^{(δ)}$. The first lemma is the key ingredient for proving the nilpotency of the radical in the case $q$ is not a root of unity.

**Lemma 7.** Suppose that $δ$ is nilpotent. Let $J$ be a nonzero $(σ, δ)$-stable ideal of $R\{x\}$ which is contained in $R$ and let $I$ be a nilpotent $σ$-stable ideal of $R^{(δ)}$. Then $Γ^m J = IΓ^m = 0$, where $γ_m ≤ 2^m − 1$ and $m$ is the index of nilpotency of $x$.

**Proof:** We claim that for any $k ∈ \{0, 1, \ldots, m\}$

$$x^{m−k} I^{γ_k} J = 0 = J I^{γ_k} x^{m−k},$$

where $γ_k ≤ 2^k − 1$.

Since $x^m = 0$, the case $k = 0$ is clear with $γ_0 = 1$. Assume that

$$x^{m−k+1} I^{γ_{k−1}} J = J I^{γ_{k−1}} x^{m−k+1} = 0,$$

where $0 ≤ k − 1 < m$ and $γ_{k−1} ≤ 2^{k−1} − 1$. Let us define the map $T : R → R\{x\}$ by setting

$$T(r) = \sum_{l=0}^{m−k} q^{−(m−k)} x^{m−k−l} σ^l(r) x^l.$$

Since $J$ is a $σ$-stable ideal of $R\{x\}$ contained in $R$, $T(J) ⊆ J$. Now, using $xa = σ(a)x$ for $a ∈ I$, $σ(x) = q^{−1}x$, (8) and the $σ$-stability of $I$ and $J$, we get:

$$δ(a T(r) b) = x(a T(r) b) − σ(a T(r) b)x$$

$$= \sum_{l=0}^{m−k} q^{−(m−k)} σ(a) x^{m−k−l+1} σ^l(r) x^l b −$$

$$− \sum_{l=0}^{m−k} q^{−(m−k)(l+1)} σ(a) x^{m−k−l} σ^{l+1}(r) x^{l+1} b$$

$$= σ(a)x^{m−k+1} rb − q^{−(m−k)(m−k+1)} σ(a)σ^{m−k+1}(r)x^{m−k+1} b$$

$$\subseteq Γ^{k−1} x^{m−k+1} J Γ^{k−1} + Γ^{k−1} J x^{m−k+1} Γ^{k−1} = 0.$$

for any $a, b ∈ Γ^{k−1}$ and $r ∈ J$. The above shows that

$$A = Γ^{k−1} T(J) Γ^{k−1} ⊆ R^{(δ)}.$$

In particular we have $IA, AI ⊆ I$, as $I$ is an ideal of $R^{(δ)}$. Let $t$ denote the index of nilpotency of the ideal $I$. Let $i_l ∈ I$, $a_l, b_l ∈ Γ^{k−1}$, $r_l ∈ J$, where $l ∈ \{1, 2, \ldots, t\}$. By the foregoing we have:

$$0 = \prod_{l=1}^t (i_l a_l T(r_l) b_l) = \prod_{l=1}^t (i_l a_l x_{m−k} r_l b_l) + "terms with x at the end".$$

Therefore, using the inductive assumption (8) we obtain:

$$\left(\prod_{l=1}^t i_l a_l x_{m−k} r_l b_l\right) x^{m−k} = 0.$$
and
\[
\prod_{l=1}^{t} x^{m-k} \sigma^{k-m}(i_l a_l r_l b_l) x^{m-k} = 0,
\]
since for \(a \in I\) we have \(xa = \sigma(a)x\). Thus
\[
(x^{m-k} \sigma^{k-m}(i_1) \sigma^{k-m}(a_1) r_1) (x^{m-k} \sigma^{k-m}(b_1) \sigma^{k-m}(i_2) \sigma^{k-m}(a_2) r_2) \ldots
\]
This implies that \(B = x^{m-k} I^{\gamma_k-1} J = x^{m-k} I^{2\gamma_k-1+1} J\) is nilpotent.

Notice that \(B\) is a right \((\sigma, \delta)\)-stable ideal of \(R\), as \(I, J \subseteq R\) are \((\sigma, \delta)\)-stable and \(J\) is an ideal of \(R\{x\}\). Since \(R\) is \((\sigma, \delta)\)-semiprime, we obtain \(0 = B = x^{m-k} I^{2\gamma_k-1+1} J\). Now, putting \(\gamma_k = 2\gamma_k-1 + 1 \leq 2^{k-1} - 1 + 1 \leq 2^k - 1\) one gets the first part of (7). The proof of the second equality is similar to the above and the lemma follows from (7) applied to \(k = m\).

**Proposition 8.** Suppose that \(R\) is a \((\sigma, \delta)\)-semiprime \(K\)-algebra and \(\delta\) is a nilpotent \(q\)-skew \(\sigma\)-derivation of \(R\) of index of nilpotency \(n\). Let \(I\) be a nilpotent \(\sigma\)-stable ideal of \(R(\delta)\). Then \(I^\gamma = 0\) for some \(\gamma \leq 2^n - 1\).

**Proof:** We will work inside the algebra \(R\{x\}\). Recall that \(x\) is nilpotent of index \(m\) which is not greater than the index of nilpotency of \(\delta\). In particular, for any nonzero ideal \(J\) of \(R\) there exists \(k\) such that \(l.\text{ann}_J(x^k) \neq 0\). The proof will be divided into three cases depending on \(q\) and the characteristic of the base field \(K\).

**Case 1.** Suppose that \(q\) is not a root of unity or \(q = 1\) and \(\text{char}K = 0\). Then, by Corollary 5, the left annihilator \(l.\text{ann}_J(x)\) is nonzero for any nonzero \((\sigma, \delta)\)-stable ideal \(J\) of \(R\). \(R\) is \((\sigma, \delta)\)-semiprime, so this implies that the ideal \(J = l.\text{ann}_R(x)\) is \((\sigma, \delta)\)-essential, i.e. it has nonzero intersection with every nonzero \((\sigma, \delta)\)-stable ideal of \(R\). Moreover, as \(J\) is \((\sigma, \delta)\)-stable, it is easy to check that \(xJ, Jx \subseteq J\). Hence Lemma 7 yields the thesis in this case.

**Case 2.** \(\text{char}K = 0\) and \(q\) is a primitive \(N\)-th root of unity, where \(N \geq 2\). Recall that, by Corollary 5, for every nonzero \((\sigma, \delta)\)-stable ideal \(J\) of \(R\), the smallest \(k\) such that \(l.\text{ann}_J(x^k) \neq 0\) is either equal to 1 or to \(N\). Let us set \(J_1 = l.\text{ann}_R(x) \cdot R, A_1 = l.\text{ann}_R(J_1)\) and \(J_2 = l.\text{ann}_{A_1}(x^N) \cdot R\). Then one can check that:

\begin{enumerate}
\item \(J = J_1 \oplus J_2\) is \((\sigma, \delta)\)-essential,
\item \(xJ_1, J_1x \subseteq J_1, x^N J_2, J_2x^N \subseteq J_2,\)
\item \(l.\text{ann}_{J_2}(x) = 0\).
\end{enumerate}

Let \(I\) be a nilpotent \(\sigma\)-stable ideal of \(R(\delta)\). Then, by Lemma 7,
\[
I^{2m-1} J_1 = J_1 I^{2m-1} = 0.
\]
We claim that for every \(j \geq 0\)
\[
x^{jN} I^{\alpha_j} J_2 = 0 = J_2 I^{\alpha_j} x^{jN},
\]
where \(\alpha_j \leq 2\alpha_{j+1} + 1\) and \(\alpha_j = 1\) for \(j \geq \frac{m}{N}\). If \(j \geq \frac{m}{N}\) then the above is clear as \(x^m = 0\). Suppose that \(j < \frac{m}{N}\) and
\[
x^{(j+1)N} I^{\alpha_{j+1}} J_2 = 0 = J_2 I^{\alpha_{j+1}} x^{(j+1)N}.
\]
Let us define \(T: R \to R\{x\}\) by setting \(T(r) = \sum_{l=0}^{w} q^{-wl} x^{wl} \sigma^l(r)x^l\), where \(w = (j+1)N - 1\). Similarly as in the proof of Lemma 7, using the inductive assumption (11), one can check that \(\delta(aT(r)b) = 0\) for any \(a, b \in I^{\alpha_{j+1}}\) and \(r \in J\). Moreover, since \(x^N J_2, J_2 x^N \subseteq J_2, \text{Lemma 6}\)
implies that $T(J_2) \subseteq J_2$. Thus $I^{\alpha_j+1}T(J_2)I^{\alpha_j+1} \subseteq J_2^{(\delta)}$ and consequently $Il^{\alpha_j+1}T(J_2)I^{\alpha_j+1} \subseteq I$. Therefore, because $I$ is nilpotent, there is $s \in \mathbb{N}$ such that $(I^{\alpha_j+1}T(J_2)I^{\alpha_j+1})^s = 0$. It means that

$$I^{\alpha_j+1}T(J_2)I^{2\alpha_j+1}T(J_2)I^{2\alpha_j+1+1} \ldots I^{2\alpha_j+1+1}T(J_2)I^{\alpha_j+1} = 0.$$  

Multiplying the above formula by $x(J+1)N-1$ on the right and applying similar arguments as in the proof of Lemma 7 one can check that $B = x(J+1)N-1I^{2\alpha_j+1+1}J_2$ is nilpotent.

By using (11) and $(\sigma, \delta)$-stability of $I$ and $J_2$ one can see that $B$ is a right $(\sigma, \delta)$-stable ideal of $R\{x\}$. Now, $(\sigma, \delta)$-semiprimeness of $R\{x\}$ gives us $B = 0$. Therefore, since $x^NI_2 \subseteq J_2$, $x^NI_2^{2\alpha_j+1+1}J_2 \subseteq \text{ann}_J(x^{N-1})$.

If $x^NI_2^{2\alpha_j+1}J_2 \neq 0$, then by Lemma 3(2) the left annihilator $\text{ann}_J(x^{N-1})$ would be also nonzero and consequently, by Corollary 5, $\text{ann}_J(x)$ is nilpotent. By the construction of $J_2$ this is impossible. Hence we obtain $x^NI_2^{2\alpha_j+1}J_2 = 0$. Now, setting $\alpha_j = 2\alpha_j+1$ one gets the first part of the statement (10). The proof of the second equality from (10) is similar.

Applying (10) for $j = 0$ we obtain $I^sJ_2 = 0$ where $\alpha \leq 2[N^2] - 1$. This together with (9) show that $I^{2m-1}J = 0$ and the proposition follows in this case, since the left annihilator of $J = J_1 \oplus J_2$ in $R$ is equal to zero and $m$ is not greater than the degree of nilpotency of $\delta$.

**Case 3.** $\text{char}K = p > 0$ and $q$ is a primitive $N$-th root of unity, where $N \geq 1$. Recall that, by Corollary 5, for every nonzero $(\sigma, \delta)$-stable ideal $J$ of $R$, the smallest $k$ such that $\text{ann}_J(x^k) \neq 0$ belongs to the set \{1, $N, Np, Np^2, \ldots$\}.

Let us set $J_0 = \text{ann}_R(x) \cdot R$, $A_0 = \text{ann}_R(J_0)$. For $i \geq 1$ define $J_i = \text{ann}_{A_{i-1}}(x^{Np^i-1}) \cdot R$ and $A_i = \text{ann}_R(J_0 + J_1 + \ldots + J_i)$. Let $s \in \mathbb{N}$ be such that $Np^{s-1} \geq m$. Then $J_s = A_{s-1} = \text{ann}_R(J_0 + J_1 + \ldots + J_{s-1})$ and $(\sigma, \delta)$-semiprimeness of $R$ implies $J_{s+i} = 0$ for all $i \in \mathbb{N}$. Moreover it is easy to check that

(i) $J = J_0 \oplus J_1 \oplus \ldots \oplus J_s$ is $(\sigma, \delta)$-essential in $R$ (possibly some of $J_i$’s are equal to 0),

(ii) $xJ_0, J_0x \subseteq J_0$ and $x^{Np^i}J_{i+1}, J_{i+1}x^{Np^i} \subseteq J_{i+1}$ for all $0 \leq i < s$,

(iii) $\text{ann}_J(x) = 0$ and $\text{ann}_J(x^{Np^i-1}) = 0$ for all $1 \leq i < s$.

Let $I$ be a nilpotent $\sigma$-stable ideal of $R^{(\delta)}$. Then, by Lemma 7,

$$I^{2m-1}J_0 = J_0I^{2m-1} = 0.$$  

We claim that for a fixed $1 \leq t \leq s$ and every $j \geq 0$

$$x^{Np^t-j}\alpha_jJ_t = 0 = J_tI^{\alpha_j}x^{Np^t-1},$$

where $\alpha_j \leq 2\alpha_j+1$ and $\alpha_j = 1$ for $j \geq \frac{m}{Np^t-\sigma}$.

Note also that for $t = 1$, the above statement was proved in Case 2. If $j \geq \frac{m}{Np^t-\sigma}$, then (13) is clear as $x^m = 0$. Suppose that $0 \leq j < \frac{m}{Np^t-\sigma}$ and

$$x^{(j+1)Np^t-1}\alpha_j+1J_t = 0 = J_tI^{\alpha_j+1}x^{(j+1)Np^t-1}.$$  

Let us define $T: R \rightarrow R\{x\}$ by setting $T(r) = \sum_{t=0}^{w} q^{-wd}x^w\sigma^t(r)x^t$, where $w = (j+1)Np^t-1$.

Similarly as in the proof of Lemma 7, using the inductive assumption (14), one can check that $\delta(aT(r)b) = 0$ for any $a, b \in I^{\alpha_j+1}$ and $r \in J_t$. Moreover, since $J_tx^{Np^t-1} \subseteq J_t$, Lemma 6 implies that $T(J_t) \subseteq J_t$. Thus $I^{\alpha_j+1}T(J_t)I^{\alpha_j+1} \subseteq J_t^{(\delta)}$ and consequently $II^{\alpha_j+1}T(J_t)I^{\alpha_j+1} \subseteq I$ is nilpotent. Using $x^{(j+1)Np^t-1-1}$ instead of $x^{(j+1)N-1}$ we can, similarly as in Case 2, conclude that the set $B = x^{(j+1)Np^t-1-1}I^{2\alpha_j+1+1}J_t$ is nilpotent.
Making use of (14) and \((\sigma, \delta)\)-stability of \(I\) and \(J_t\) one can see that \(B\) is a right \((\sigma, \delta)\)-stable ideal of \(R\{x\}\) and, as in Case 2, \(B = 0\) follows. Then, since \(x^{Np^t-1}J_t \subseteq J_t\), we obtain

\[x^{Np^t-1} I^{2\alpha_j+1} J_t \subseteq r.\text{ann}_{J_t}(x^{Np^t-1}).\]

If \(r.\text{ann}_{J_t}(x^{Np^t-1}) \neq 0\), then Lemma 3(2) and Corollary 5 imply that either \(1.\text{ann}_{J_t}(x^{Np^t-2}) \neq 0\) if \(t \geq 2\) or \(1.\text{ann}_{J_t}(x)\) if \(t = 1\). This contradicts the property (iii) of \(J_t\) and \(x^{Np^t-1} I^{2\alpha_j+1} J_t = 0\) follows. Setting \(\alpha_j = 2\alpha_j+1 + 1\) one gets the first part of the statement (13). The proof of the second equality from (13) is similar.

Now applying (13) for \(j = 0\) we obtain \(I^{\beta_j} J_t = 0\) for all \(1 \leq t \leq s\), where \(\beta_t \leq 2^{\frac{n-1}{[x:R]}}\). This together with (12) shows that \(I^{2n-1}J = 0\). Because \(J = J_0 \oplus J_1 \oplus \ldots \oplus J_s\) is \((\sigma, \delta)\)-essential, \(I^{2n-1} = 0\). This completes the proof. \(\square\)

Now, with all the above preparation, we are ready to prove our main result.

**Theorem 9.** Let \(R\) be a \((\sigma, \delta)\)-semiprime \(K\)-algebra acted by an algebraic \(q\)-skew \(\sigma\)-derivation \(\delta\) of \(R\) which satisﬁes a polynomial over \(K\) of degree \(n\). Then:

1. If \(I\) is a nilpotent \(\sigma\)-stable ideal of \(R(\delta)\) then \(I^n = 0\), for some \(\gamma \leq 2^n - 1\).
2. \(R(\delta)\) contains the largest nilpotent \(\sigma\)-stable ideal \(B_\sigma(R(\delta))\) such that \(B_\sigma(R(\delta)/B_\sigma(R(\delta))) = 0\), i.e. \(B_\sigma(R(\delta))\) is the \(\sigma\)-prime radical of \(R(\delta)\). The nilpotency index of \(B_\sigma(R(\delta))\) is bounded by \(2^n - 1\).
3. If in addition, \(\sigma\) is algebraic on \(R(\delta)\), then the prime radical \(B(R(\delta))\) of \(R(\delta)\) is nilpotent of index bounded by \(2^n - 1\).

**Proof:** Let \(t(R)\) denote the 0-eigenspace of \(\delta\), i.e.

\[t(R) = \{ r \in R \mid \delta^k(r) = 0 \text{ for some } k \geq 1 \}\]

By [1, Theorem 6], \(t(R)\) is an \((\sigma, \delta)\)-semiprime subalgebra of \(R\). Clearly \(\delta\) is a nilpotent \(q\)-skew \(\sigma\)-derivation of \(t(R)\) of nilpotency index not greater than \(n\) and \(t(R(\delta)) = R(\delta)\). Now the first statement of the theorem is a direct consequence of Proposition 8. (2) is a standard application of (1). Suppose additionally that \(\sigma\) is algebraic on \(R(\delta)\). Then every nilpotent ideal of \(R(\delta)\) is contained in a nilpotent ideal which is \(\sigma\)-stable. Now (3) is a consequence of (2). \(\square\)

Note that even when \(\sigma = \text{id}_R\), i.e. \(\delta\) is a usual derivation, then \(\delta\)-semiprime algebra does not have to be semiprime. Thus the above theorem generalizes [5, Theorem 3.3] even in the case of derivations. On the other hand, when both \(\sigma\) and \(\delta\) are algebraic and either \(q\) is not a root of unity or \(q = 1\) and \(\text{char}K = 0\), then \((\sigma, \delta)\)-semiprimeness of \(R\) is equivalent to semiprimeness of \(R\) (cf. [6]). Let us record at the end the special case of the above theorem when \(\delta = \sigma - \text{id}_R\). Then \(R(\delta) = R(\sigma) = \{ r \in R \mid \sigma(r) = r \}\) and we have the following:

**Corollary 10.** Suppose that \(R\) is a semiprime \(K\)-algebra acted by an algebraic automorphism \(\sigma\) which satisfies a polynomial over \(K\) of degree \(n\). Then the prime radical of \(R(\sigma)\) is nilpotent and the index of nilpotency is bounded by \(2^n - 1\).

The above result was obtained in [5] under an additional assumption on the characteristic of \(K\).

**ACKNOWLEDGMENT**

This research was supported by Polish Scientific Grant KBN no. 2 PO3A 039 14.
References