ACTIONS OF LIE SUPERALGEBRAS ON SEMIPRIME ALGEBRAS WITH CENTRAL INVARIANTS

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Abstract

Let $R$ be a semiprime algebra over a field $K$ of characteristic zero acted finitely on by a finite dimensional Lie superalgebra $L = L_0 \oplus L_1$. It is shown that if $L$ is nilpotent, $[L_0, L_1] = 0$ and the subalgebra of invariants $R^L$ is central, then the action of $L_0$ on $R$ is trivial and $R$ satisfies the standard polynomial identity of degree $2 \cdot \sqrt{\dim_K L_1}$. Examples of actions of nilpotent Lie superalgebras with central invariants and with $[L_0, L_1] \neq 0$, are constructed.

1 Preliminaries

If $R$ is an algebra over a field $K$ of characteristic $\neq 2$ and $\sigma$ is a $K$-linear automorphism of $R$ such that $\sigma^2 = 1$, let $D_0 = \{ \delta \in \text{End}_K(R) \mid \delta(rs) = \delta(r)s + r\delta(s) \mathrm{ and } \delta\sigma(r) = \sigma\delta(r) \mathrm{ for all } r, s \in R \}$ and $D_1 = \{ \delta \in \text{End}_K(R) \mid \delta(rs) = \delta(r)s + \sigma(r)\delta(s) \mathrm{ and } \delta\sigma(r) = -\sigma\delta(r) \mathrm{ for all } r, s \in R \}$. Then $D_0 \oplus D_1$ is a Lie superalgebra and the elements of $D_0$ and $D_1$ are respectively, derivations and skew derivations of $R$. The superbracket on $D_0 \oplus D_1$ is defined as $[\delta_1, \delta_2] = \delta_1\delta_2 - (-1)^{ij}\delta_2\delta_1$, where $\delta_1 \in D_i$, $\delta_2 \in D_j$ and $i, j \in \{0, 1\}$. If $L = L_0 \oplus L_1$ is a Lie superalgebra, we say that $L$ acts on $R$ if there is a homomorphism $\psi$ of $L$ to $D_0 \oplus D_1$ identifying the elements of $L_0$ and $L_1$ with their images under $\psi$. It is well known that the homomorphism $\psi$ induces an associative homomorphism from the universal enveloping algebra $U(L)$ to $\text{End}_K(R)$ and its image is finite dimensional if and only if the derivations and skew derivations from $L_0$ and $L_1$ are algebraic. In this case we will say that $L$ acts finitely on $R$. Letting $G$ be the group $\{1, \sigma\}$, we can form the skew group ring $H = U(L) * G$ and $H$ is now a Hopf algebra acting on $R$. When $L$ acts on $R$, we define the subalgebra of invariants...
$R^L$ to be the set $\{ r \in R \mid \delta(r) = 0, \text{ for all } \delta \in L \}$. Depending upon the context, the symbol $[,]$ may represent either the superbracket on $L$, or the commutator map $[r, s] = rs - sr$, where $r, s$ belong to an associative algebra. Inductively, we let $L^1 = L$ and $L^{n+1} = [L^n, L]$ and we say that $L$ is nilpotent if there exists a positive integer $N$ such that $L^N = 0$. If $R$ (resp. $L$) is an associative algebra (resp. Lie superalgebra) we will let $Z(R)$ (resp. $Z(L)$) denote its center. For an element $a \in R$, and automorphism $\sigma$ of $R$, $\text{ad}_a$ (resp. $\partial_a$) stands for the inner derivation (inner $\sigma$-derivation) adjoint to $a$, i.e., $\text{ad}_a(x) = ax - xa$ ($\partial_a(x) = ax - \sigma(x)a$).

2 Main result

The main aim of this paper is to prove the following theorem.

**Theorem 1.** Let a finite dimensional nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts finitely on a semiprime $\mathbb{K}$-algebra $R$, where $\mathbb{K}$ is a field of characteristic zero. If $R^L$ is central and $[L_1, L_0] = 0$, then $R$ satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2\dim_K L_1}]$.

It generalizes a result from [1] concerning the actions of nilpotent Lie algebras of characteristic zero on semiprime algebras. On the other hand, in [4] it is proved that if a pointed Hopf algebra $H$ acts finitely of dimension $N$ on a semiprime algebra $R$ and the action is such that $L^H \neq 0$ for any nonzero $H$-stable left ideal $L$ of $R$ and $R^H \subseteq Z(R)$, then $R$ satisfies PI of degree $2[\sqrt{N}]$. In Theorem 1 we prove for nilpotent Lie superalgebras with $[L_0, L_1] = 0$, that the dimension of the action of $U(L) \ast G$ depends only on the dimension of $L_1$. The key role will be played by the following easy observation: In characteristic zero the invariants of nilpotent Lie algebras acting on central simple algebras are never proper simple central subalgebras.

**Lemma 2.** Let $R$ be a finite dimensional central simple $\mathbb{F}$-algebra acted on by a nilpotent Lie $\mathbb{F}$-algebra $L$, where $\mathbb{F}$ is a field of characteristic zero. If $R^L$ is a central simple $\mathbb{F}$-algebra, then $R = R^L$. In this case the action of $L$ on $R$ must be trivial.

**Proof:** Since $L$ acts by $\mathbb{F}$-linear transformations, any derivation from $L$ is inner. Suppose that the action of $L$ on $R$ is not trivial. Then we can take a nonzero derivation $\delta = \text{ad}_a \in Z(L)$, where $a \in R$. For any $a_b \in L$ we have $\text{ad}_{[a, b]} = [\text{ad}_a, \text{ad}_b] = 0$, so $[a, b] \in Z(R) = \mathbb{F}$. If $[a, b] = \lambda \neq 0$, then $[a, \lambda^{-1}b] = 1$. Note that the elements $a$ and $\lambda^{-1}b$ generate in $R$ a subalgebra isomorphic to the Weyl algebra $A_1(\mathbb{F})$, but it is impossible since $R$ is finite dimensional. Consequently, $[a, b] = 0$ for any $a_b \in L$ and hence $a \in R^L$. In particular, $\text{ad}_a$ acts trivially on $C_R(R^L)$, the centralizer of $R^L$ in $R$. On the other hand the subalgebra $R^L$ is simple and $Z(R^L) = \mathbb{F}$, so by Theorem 1 (p. 118) in [5] $R \simeq R^L \otimes_{\mathbb{F}} C_R(R^L) \simeq R^L \cdot C_R(R^L)$. Consequently, $R = R^L \cdot C_R(R^L)$. It implies that $\text{ad}_a$ acts trivially on $R$, a contradiction. Therefore the action of $L$ on $R$ is trivial. $\square$

Suppose that a finite dimensional nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts finitely of dimension $N$ on an algebra $R$. Then by $R^L_0$ we denote the largest subspace
of $R$ on which any derivation from $L_0$ acts nilpotently, that is

$$R_{L_0} = \{ r \in R \mid \delta^N(r) = 0, \; \forall \delta \in L_0 \}.$$  

It is clear that $R_{L_0}$ is a subalgebra of $R$ and $R_{L_0}$ is stable under the automorphism $\sigma$. Furthermore, it is well known that (after eventual extension of the field of scalars) the algebra $R$ is graded (with finite support) by the dual of the Lie algebra $L_0$ with $R_{L_0}$ as the identity component of the grading. Therefore, if the algebra $R$ is semiprime (semisimple), then $R_{L_0}$ is also semiprime (resp. semisimple). In [3] (Lemma 12) it is proved that

**Lemma 3.** The subalgebra $R_{L_0}$ is $L$-stable. In particular $L$ acts on $R_{L_0}$ by nilpotent transformations.

In the next Proposition we consider the case of action of a nilpotent Lie superalgebra on a finite dimensional $G$-simple algebra.

**Proposition 4.** Let a nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts on a $G$-simple finite dimensional $\mathbb{K}$-algebra $R$, where $\mathbb{K}$ is a field of characteristic zero. If $R^L$ is central and $[L_0, L_1] = 0$, then $L_0 = 0$.

**Proof:** First we will consider the case when $L$ acts on $R$ by nilpotent transformations, that is $R = R_{L_0}$. Suppose that $L_0 \neq 0$ and take a nonzero derivation $\delta$ from the center of $L_0$. Since $[L_0, L_1] = 0$, it is clear that $\delta$ is in the center of $L$. Let $k > 1$ be such that $\delta^k(R) = 0$ and $V = \delta^{k-1}(R) \neq 0$. Then $V$ is invariant under the action of $L$, and since $L$ acts via nilpotent transformations it is clear that $V^L = V \cap R^L \subseteq Z(R)$. On the other hand if $r, s \in R$, then the Leibniz rule gives

$$0 = \delta^k(\delta^{k-2}(r)s) = k\delta^{k-1}(r)\delta^{k-1}(s).$$

It means that $(V^L)^2 = 0$, so the center of $R$ contains nilpotent elements. This is impossible since $R$ is semisimple. The obtained contradiction shows that $L_0 = 0$.

Consider the general case. The above gives us immediately that $R^L = R_{L_0}$ and consequently the algebra $R_{L_0}$ is semisimple. Thus, any its ideal $I$ is idempotent, i.e., $I^2 = I$. Note that if $I$ is $G$-stable, then the Leibniz rule, applied to any $\partial \in L_1$, gives $\partial(I) = \partial(I^2) \subseteq \partial(I) + \sigma(I)\partial(I) \subseteq I$. Hence any $G$-stable ideal $I$ of $R_{L_0}$ is also $L$-stable and $0 \neq I^L \subseteq Z(R)$. Thus $I$ contains invertible elements. Consequently, $R_{L_0}$ is also $G$-simple.

We will split considerations into two cases. First suppose that the automorphism $\sigma$ is inner, and let $q \in R$ be such that $\sigma(x) = q^{-1}xq$, for $x \in R$. In this case any ideal of $R$ is $\sigma$-stable, so $R$ must be a simple algebra. Moreover it is easy to see that any skew derivation $\partial$ from $L_1$ must be inner. Indeed, since $\partial \sigma = -\sigma \partial$, we obtain that

$$q^{-1}\partial(x)q = \sigma(\partial(x)) = -\partial(\sigma(x)) = -\partial(q^{-1}xq) =$$

$$= -\partial(q^{-1})xq - q^{-1}\partial(x)q - q^{-1}\sigma(x)\partial(q).$$

Since $q\partial(q^{-1}) = -\partial(q)q^{-1}$,

$$\partial(x) = -\frac{1}{2}q\partial(q^{-1})x - \frac{1}{2}\sigma(x)\partial(q)q^{-1} = \frac{1}{2}q\partial(q^{-1})x - \sigma(x)\frac{1}{2}\partial(q)q^{-1}.$$
This immediately gives, that \( \partial(x) = bx - \sigma(x)b \), where \( b = \frac{1}{2} \partial(q)q^{-1} \). Consequently, any mapping from \( L_0 \cup L_1 \) is \( Z(R) \)-linear. We will show that the algebra \( R^{L_0} \) is simple and the centers of \( R^{L_0} \) and \( R \) coincide. Since the automorphism \( \sigma \) has order two, \( q^2 \in Z(R) \). Thus for any \( \delta = \text{ad}_a \in L_0 \),
\[
\delta(q) = \delta(\sigma(q)) = \sigma(\delta(q)) = q^{-1}(aq - qa)q = qa - aq = -\delta(q),
\]
so \( \delta(q) = 0 \). This implies that \( q \in R^{L_0} \), the restriction of \( \sigma \) to \( R^{L_0} \) is inner and hence the algebra \( R^{L_0} \) is simple. Since the action of \( L \) on \( R \) is inner, \( Z(R) = Z(R) \cap R^{L_0} \subseteq Z(R^{L_0}) \). We will show that \( Z(R^{L_0}) \subseteq Z(R) \). To this end, since \( R^L \subseteq Z(R) \), it suffices to show that \( Z(R^{L_0}) \subseteq R^{L_1} \). Take any \( z \in Z(R^{L_0}) \), and \( \partial = \partial_b \in L_1 \), where \( b = \frac{1}{2} \partial(q)q^{-1} \). Notice that \( b \in R^{L_0} \). Indeed, by assumption \([\delta, \partial] = 0\) for any \( \delta \in L_0 \) and by the above \( q \in R^{L_0} \), so
\[
\delta(b) = \frac{1}{2} \delta(\partial(q)q^{-1}) = \frac{1}{2} \delta(\partial(q))q^{-1} + \frac{1}{2} \partial(\delta(q))q^{-1} = \frac{1}{2} \partial(\delta(q))q^{-1} = 0.
\]
It means that \( b \in R^{L_0} \) and
\[
\partial(z) = bz - \sigma(z)b = bz - zb = 0,
\]
so \( z \in R^{L_1} \). It proves that \( Z(R^{L_0}) = Z(R) \). By Lemma 2 the action of \( L_0 \) on \( R \) must be trivial.

Finally suppose that the automorphism \( \sigma \) is outer. Since \( R \) is \( G \)-simple, the algebra \( R \) must be either simple or \( R = I \oplus \sigma(I) \) for some minimal ideal \( I \). In the first case, by the Skolem-Noether Theorem, \( \sigma \) is not an identity map on \( Z(R) \). In the second case \( Z(R) = Z(I) \oplus \sigma(Z(I)) \). Thus in both cases \( \sigma \) acts non identically on \( Z(R) \). Now since the center of \( R^{L_0} \) contains \( Z(R) \), the restriction of \( \sigma \) to \( R^{L_0} \) is also outer. Consequently, one can choose a nonzero element \( c \in Z(R) \) such that \( \sigma(c) \neq c \). Then \((c - \sigma(c))^2\) is nonzero and belongs to the field \( Z(R)^\sigma \). Thus \( c - \sigma(c) \) is invertible. Now let \( \partial \in L_1 \) and \( x \in R \). Notice that
\[
\partial(x)c + \sigma(x)\partial(c) = \partial(xc) = \partial(cx) = \partial(c)x + \sigma(c)\partial(x).
\]
In particular, we have
\[
\partial(x) = (c - \sigma(c))^{-1}\partial(c)x - \sigma(x)(c - \sigma(c))^{-1}\partial(c) = \partial_b(x),
\]
where \( b = (c - \sigma(c))^{-1}\partial(c) \). Thus \( L_1 \) acts on \( R \) via inner \( \sigma \)-derivations and in particular every mapping from \( L \) is \( Z(R)^\sigma \)-linear. We will prove that \( Z(R^{L_0})^\sigma = Z(R)^\sigma \). Similarly as above, it suffices to show that \( Z(R^{L_0})^\sigma \subseteq R^{L_1} \). Take any \( \partial = \partial_b \in L_1 \), where \( b = (c - \sigma(c))^{-1}\partial(c) \) for some \( c \in Z(R) \). Since \( L_0 \) acts trivially on the center of \( R \), one obtains that \( b \in R^{L_0} \). Now it is clear that \( \partial_b \) acts trivially on \( Z(R^{L_0})^\sigma \), and consequently \( Z(R^{L_0})^\sigma \subseteq R^{L_1} \).

Consider skew group rings \( R \ast G \) and \( R^{L_0} \ast G \). Since both of \( R \) and \( R^{L_0} \) are \( G \)-simple, and \( \sigma \) is outer on \( R \) and \( R^{L_0} \), the rings \( R \ast G \) and \( R^{L_0} \ast G \) are simple. Moreover it is clear that \( Z(R \ast G) = Z(R)^\sigma \) and \( Z(R^{L_0} \ast G) = Z(R^{L_0})^\sigma \). Thus \( R \ast G \) and \( R^{L_0} \ast G \) are central simple \( Z(R)^\sigma \)-algebras. Notice that the action of \( L_0 \) on \( R \) can be extended to an action on \( R \ast G \), via the formula \( \delta(a + b\sigma) = \delta(a) + \delta(b)\sigma \). In that case \((R \ast G)^{L_0} = R^{L_0} \ast G \) Again applying Lemma 2 we obtain that \( L_0 \) must act trivially on \( R \) and the proof is complete. \( \square \)
Central invariants

**Corollary 5.** Let a nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts on a $G$-simple finite dimensional $\mathbb{K}$-algebra $R$ with center $Z$, where $\text{char} \mathbb{K} = 0$. If $R^L \subseteq Z$ and $[L_0, L_1] = 0$, then $\dim_{Z^G} R \leq [Z : Z^G] \cdot 2^{\dim_{Z^L} L_1}$. Moreover, in this case $R$ satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{Z^L} L_1}}]$.

**Proof:** By Proposition 4, $L_0 = 0$. Thus $L$ is spanned by a family $\{\partial_1, \ldots, \partial_n\}$ of inner skew derivations such that $\partial_j^2 = 0$ and $\partial_i \partial_j + \partial_j \partial_i = 0$. It is clear that every $\partial_j$ is $Z^G$-linear. Let us consider a chain

$$V_0 = R \supseteq V_1 \supseteq \cdots \supseteq V_n$$

of subspaces of $R$, where $V_j = \ker \partial_1 \cap \cdots \cap \ker \partial_j$ for $j = 1, \ldots, n$. Then $V_n \subseteq R^L \subseteq Z$ and $\partial_j$ maps $V_{j-1}$ into $V_j$. Moreover, it is clear that $\dim_{Z^G} V_{j-1} = \dim_{Z^G} (\ker \partial_j \cap V_{j-1}) + \dim_{Z^G} \partial_j(V_{j-1}) \leq 2 \cdot \dim_{Z^G} V_j$. Thus

$$\dim_{Z^G} R \leq 2^n \cdot \dim_{Z^G} V_n \leq [Z : Z^G] \cdot 2^{\dim_{Z^L} L_1}.$$ 

Since $R$ is $G$-simple, the algebra $R$ must be either simple or $R = I \oplus \sigma(I)$ for a minimal ideal $I$ of $R$. Then $I$ is certainly a simple algebra. The above inequality implies that $\dim_{Z^G} R \leq 2^{\dim_{Z^L} L_1}$ in the first case, and $\dim_{Z^G} I \leq 2^{\dim_{Z^L} L_1}$ in the second case. The result follows now by the Amitsur-Levitzki Theorem.

If $R$ is semiprime we let $Q = Q(R)$ to denote the symmetric Martindale quotient ring. Its center, known as the extended centroid of $R$, we denote by $C$. The following properties of $Q$ in the case when $R$ is acted on by a Hopf algebra are proved in [3].

**Lemma 6.** Let $R$ be a semiprime $H$-module algebra such that the $H$-action on $R$ extends to an $H$-action on $Q$ and any nonzero $H$-stable ideal of $R$ contains nontrivial invariants. Then

1. the ring $C^H = C \cap Q^H$ is von Neumann regular and selfinjective.
2. If a nonempty subset $S \subseteq C^H \setminus \{0\}$ is closed under a multiplication, then the localization $Q_S$ of $Q$ at $S$ is semiprime and $Z(Q_S) = C_S$.
3. If $H$ acts finitely on $Q$ and $S = C^H \setminus M$, where $M$ is a maximal ideal of $C^H$, then the $H$-action on $Q$ extends to an $H$-action on $Q_S$ and $(Q^H)_S = (Q_S)^H$, $(C^H)_S = (C^H)_S = C_S \cap (Q_S)^H$ is a field contained in the center of $Q_S$.

We can now prove the main result of the paper.

**Proof of Theorem 1:** Let $H = U(L) \ast G$. By ([2], Corollary 6) every $H$-invariant non-nilpotent subalgebra of $R$ contains nonzero invariants. Thus we can apply the results from [4]. Let $M$ be a maximal ideal of $C^H = C \cap Q^H$ and put $S = C^H \setminus M$. By the above lemma and [4] it follows that $(C_S)^H$ is a field and $Q_S$ is a finite dimensional, $G$-simple $(C_S)^H$-algebra. Using Corollary 5 we obtain that $Q_M$ satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{Z^L} L_1}}]$. Since it holds for any maximal ideal $M$ of $C^H$, the ring $Q$, and consequently $R$, satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{Z^L} L_1}}]$. 

3 Examples

In this section we construct examples of actions of nilpotent Lie superalgebras with central invariants and with $[L_0, L_1] \neq 0$. We start with general properties of inner derivations and skew derivations of an algebra $R$ with an automorphism $\sigma$ of order two. Then $R = R_0 \oplus R_1$ is $\mathbb{Z}_2$-graded, where $R_0 = \{ x \in R \mid \sigma(x) = x \}$ and $R_1 = \{ x \in R \mid \sigma(x) = -x \}$. For any inner derivation $\delta$ of $R$, the condition $\delta \sigma = \sigma \delta$ is equivalent to that $\delta$ is induced by some $a \in R_0$. To see that, we let $\delta$ be induced by $a = a_0 + a_1 \in R$. Then

$$\delta(x) = ax - xa = (a_0 x - xa_0) + (a_1 x - xa_1).$$

This immediately implies that

$$\delta(\sigma(x)) = (a_0 \sigma(x) - \sigma(x)a_0) + (a_1 \sigma(x) - \sigma(x)a_1)$$

and

$$\sigma(\delta(x)) = (a_0 \sigma(x) - \sigma(x)a_0) - (a_1 \sigma(x) - \sigma(x)a_1).$$

Since $\delta$ and $\sigma$ commute, the previous equations imply that $a_1 \sigma(x) - \sigma(x)a_1 = 0$. Replacing $x$ by $\sigma(x)$ yields $a_1 x - xa_1 = 0$. Equation (1) now becomes

$$\delta(x) = a_0 x - xa_0 = ad_{a_0}(x).$$

In the same manner we can show that for any inner skew derivation $\partial$ of $R$, the condition $\partial \sigma = -\sigma \partial$ is equivalent to that $\partial = \partial_b$ for some $b \in R_1$.

**Lemma 7.** Let $R$ be an algebra over a field $\mathbb{K}$ of characteristic $\neq 2$ and $\sigma$ be a $\mathbb{K}$-linear automorphism of $R$ of order $2$. Let $u \in R$ be invertible and $\sigma(u) = -u$. Let $\tilde{R}$ be the $\mathbb{K}$-algebra $M_2(R)$, the $2 \times 2$ matrices over $R$. Then the map $\varphi: R \to \tilde{R}$ given by

$$\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & u^{-1}\sigma(x)u \end{pmatrix}$$

is an injective homomorphism of algebras, satisfying $\tilde{\sigma} \varphi = \varphi \sigma$ (where $\tilde{\sigma}$ is a componentwise extension of $\sigma$ to $\tilde{R}$).

If a Lie superalgebra $L = L_0 \oplus L_1$ acts on $R$ by inner derivations and inner $\sigma$-derivations with $R^L = \mathbb{K}$, then $L$ acts on $\tilde{R}$ by inner derivations and inner $\tilde{\sigma}$-derivations with

$$\tilde{R}^L = \left\{ \begin{pmatrix} \alpha & \beta u \\ \gamma u^{-1} & \lambda \end{pmatrix} \in \tilde{R} \mid \alpha, \beta, \gamma, \lambda \in \mathbb{K} \right\}.$$

**Proof:** Notice that

$$\tilde{\sigma}(\varphi(x)) = \tilde{\sigma}\left( \begin{pmatrix} x & 0 \\ 0 & u^{-1}\sigma(x)u \end{pmatrix} \right) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & u^{-1}xu \end{pmatrix} = (\varphi \sigma)(x).$$

In order to prove the second part, observe that for all inner derivation $ad_a \in L_0$ and the inner skew derivation $\partial_b \in L_1$ of $R$ and for every matrix $\tilde{x} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \tilde{R}$ the following equations hold

$$\text{ad}_{\varphi(a)}(\tilde{x}) = \begin{pmatrix} a & 0 \\ 0 & u^{-1}au \end{pmatrix} \cdot \tilde{x} - \tilde{x} \cdot \begin{pmatrix} a & 0 \\ 0 & u^{-1}au \end{pmatrix} = \begin{pmatrix} \text{ad}_a(x_{11}) & \text{ad}_a(x_{12}u^{-1})u \\ u^{-1}\text{ad}_a(u_{x_{21}}) & u^{-1}\text{ad}_a(u_{x_{22}}u^{-1})u \end{pmatrix},$$

$$\text{ad}_{\varphi(b)}(\tilde{x}) = \begin{pmatrix} 0 & u^{-1}bu \\ \gamma u^{-1} & \lambda \end{pmatrix} \cdot \tilde{x} - \tilde{x} \cdot \begin{pmatrix} 0 & u^{-1}bu \\ \gamma u^{-1} & \lambda \end{pmatrix} = \begin{pmatrix} \text{ad}_b(x_{11}) & \text{ad}_b(x_{12}u^{-1})u \\ u^{-1}\text{ad}_b(u_{x_{21}}) & u^{-1}\text{ad}_b(u_{x_{22}}u^{-1})u \end{pmatrix}. $$
and
\[
\partial_{\varphi(b)}(\tilde{x}) = \begin{pmatrix} b & 0 \\ 0 & -u^{-1}bu \end{pmatrix} \cdot \tilde{x} - \bar{\sigma}(\tilde{x}) \cdot \begin{pmatrix} b & 0 \\ 0 & -u^{-1}bu \end{pmatrix} = \\
= \begin{pmatrix} \partial_b(x_{11}) & \partial_b(x_{12}u^{-1})u \\ \sigma(u^{-1})\partial_b(ux_{21}) & \sigma(u^{-1})\partial_b(ux_{22}u^{-1})u \end{pmatrix}.
\]

¿From the above equations it follows that \( \tilde{x} \in \tilde{R}^L \) if and only if the elements \( x_{11}, x_{12}u^{-1}, ux_{21} \) and \( ux_{22}u^{-1} \) belong to \( R^L \). Under the assumption that \( R^L = \mathbb{K} \), we now easily obtain the assertion of the lemma.

We start our construction from the algebra \( R = M_2(\mathbb{K}) \) of \( 2 \times 2 \) matrices over a field \( \mathbb{K} \) of characteristic 0. Let \( \sigma \) be the inner automorphism of order 2 of \( R \) induced by the diagonal matrix \( \text{diag}(1, -1) \) and let \( \partial_{b_1} \) and \( \partial_{b_2} \) be the inner \( \sigma \)-derivations of \( R \) induced by
\[
b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in R_1 \quad \text{and} \quad b_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in R_1,
\]
respectively. It can be easily checked that
\[
b_1^2 = -b_2^2 = 1 \quad \text{and} \quad b_1b_2 + b_2b_1 = 0.
\]
As a result, the skew derivations \( \partial_{b_1} \) and \( \partial_{b_2} \) span an abelian Lie superalgebra \( L = L_0 \oplus L_1 \) where \( L_0 = 0 \) and \( L_1 = \text{Span}_\mathbb{K}\{\partial_{b_1}, \partial_{b_2}\} \).

¿From the explicit formulas for \( \partial_{b_1} \) and \( \partial_{b_2} \)
\[
\partial_{b_1} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} + x_{12} & x_{22} - x_{11} \\ x_{11} - x_{22} & x_{21} + x_{12} \end{pmatrix}
\]
and
\[
\partial_{b_2} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} - x_{12} & x_{22} - x_{11} \\ x_{22} - x_{11} & x_{21} - x_{12} \end{pmatrix}
\]
it follows immediately that \( R^L = \mathbb{K} \).

Using Lemma 7, applied to the invertible element \( u = b_2 \), we have an embedding of \( R \) into \( \tilde{R} = M_2(R) \), according to the formula
\[
\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & b_2^{-1}\sigma(x)b_2 \end{pmatrix}.
\]
Put
\[
\tilde{b}_1 = \varphi(b_1) = \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} \in \tilde{R}_1 \quad \text{and} \quad \tilde{b}_2 = \varphi(b_2) = \begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix} \in \tilde{R}_1
\]
and consider the additional matrices
\[
\tilde{b}_3 = \begin{pmatrix} 0 & b_2 \\ -b_2 & 0 \end{pmatrix} \in \tilde{R}_1 \quad \text{and} \quad \tilde{b}_4 = \begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix} \in \tilde{R}_1.
\]
It is not hard to check that
\[
\tilde{b}_1^2 = -\tilde{b}_2^2 = \tilde{b}_3^2 = -\tilde{b}_4^2 = 1 \quad \text{and} \quad \tilde{b}_i\tilde{b}_j + \tilde{b}_j\tilde{b}_i = 0
\]
for all $i \neq j$. As before, the inner skew derivations $\partial_{b_1}$, $\partial_{b_2}$, $\partial_{b_3}$ and $\partial_{b_4}$ span an abelian Lie superalgebra $\tilde{L} = \tilde{L}_0 \oplus \tilde{L}_1$, where $\tilde{L}_0 = 0$ and $\tilde{L}_1 = \text{span}_K\{\partial_{b_1}, \partial_{b_2}, \partial_{b_3}, \partial_{b_4}\}$. Lemma 7 says that the subalgebra of invariants $\tilde{R}^L$ under the action of $L$ consists of all matrices of the form $\begin{pmatrix} \alpha & \beta b_2 \\ \gamma b_2 & \lambda \end{pmatrix}$, where $\alpha, \beta, \gamma, \lambda \in K$. Furthermore, a simple calculation shows that 
\[ \partial_{b_3}(\begin{pmatrix} \alpha & \beta b_2 \\ \gamma b_2 & \lambda \end{pmatrix}) = \begin{pmatrix} \beta - \gamma & (\lambda - \alpha)b_2 \\ (\lambda - \alpha)b_2 & \beta - \gamma \end{pmatrix} \]
and 
\[ \partial_{b_4}(\begin{pmatrix} \alpha & \beta b_2 \\ \gamma b_2 & \lambda \end{pmatrix}) = \begin{pmatrix} -\beta - \gamma & (\lambda - \alpha)b_2 \\ (\alpha - \lambda)b_2 & -\beta - \gamma \end{pmatrix}. \]
This immediately implies that $\tilde{R}^L = K$.

Applying Lemma 7 for the invertible element $u = \tilde{b}_4$ we have the next embedding of $\tilde{R}$ into the algebra $R = M_2(\tilde{R})$, the $2 \times 2$ matrices over $\tilde{R}$ according to the formula
\[ \varphi(\tilde{x}) = \begin{pmatrix} \tilde{x} & 0 \\ 0 & \tilde{b}_4^{-1}\tilde{x}\tilde{b}_4 \end{pmatrix}. \]
Put 
\[ B_i = \varphi(\tilde{b}_i) = \begin{pmatrix} \tilde{b}_i & 0 \\ 0 & \tilde{b}_i \end{pmatrix} \in R_1 \text{ and } B_4 = \varphi(\tilde{b}_4) = \begin{pmatrix} \tilde{b}_4 & 0 \\ 0 & -\tilde{b}_4 \end{pmatrix} \in R_1 \]
for $i = 1, 2, 3$ and consider the additional matrices

- $A_1 = \begin{pmatrix} 0 & \tilde{a}_1 \\ -\tilde{a}_1 & 0 \end{pmatrix} \in R_0$ and $C_1 = \begin{pmatrix} 0 & \tilde{a}_1 \\ 0 & 0 \end{pmatrix} \in R_0$, where $\tilde{a}_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in \tilde{R}_0$,

- $A_2 = \begin{pmatrix} 0 & \tilde{a}_2 + 1 \\ -\tilde{a}_2 + 1 & 0 \end{pmatrix} \in R_0$ and $C_2 = \begin{pmatrix} 0 & \tilde{a}_2 + 1 \\ 0 & 0 \end{pmatrix} \in R_0$, where $\tilde{a}_2 = \begin{pmatrix} b_1 b_2 & 0 \\ b_1 b_2 & 0 \end{pmatrix} \in \tilde{R}_0$,

- $A_3 = \begin{pmatrix} \tilde{a}_3 - \tilde{a}_1 & 0 \\ 0 & \tilde{a}_3 + \tilde{a}_1 \end{pmatrix} \in R_0$, where $\tilde{a}_3 = \begin{pmatrix} b_1 b_2 & b_1 b_2 \\ -b_1 b_2 & -b_1 b_2 \end{pmatrix} \in \tilde{R}_0$,

- $B_5 = \begin{pmatrix} 0 & \tilde{d}_5 \\ \tilde{d}_5 & 0 \end{pmatrix} \in R_1$, $B_6 = \begin{pmatrix} 0 & \tilde{b}_4 \\ -\tilde{b}_4 & 0 \end{pmatrix} \in R_1$ and $B_7 = \begin{pmatrix} 0 & \tilde{b}_4 \\ \tilde{b}_4 & 0 \end{pmatrix} \in R_1$, where $\tilde{d}_5 = \begin{pmatrix} b_1 + b_2 & b_1 + b_2 \\ -b_1 - b_2 & -b_1 - b_2 \end{pmatrix}$, $\tilde{b}_5 = \begin{pmatrix} -b_1 + b_2 & -b_1 + b_2 \\ b_1 - b_2 & b_1 - b_2 \end{pmatrix} \in \tilde{R}_1$,

- $D_5 = \begin{pmatrix} 0 & \tilde{d}_5 \\ 0 & 0 \end{pmatrix} + B_7 \in R_1$ and $D_6 = \begin{pmatrix} 0 & \tilde{b}_4 \\ 0 & 0 \end{pmatrix} \in R_1$.

Notice that if $N_0 = \text{span}_K\{\text{ad}_{C_1}, \text{ad}_{C_2}, \text{ad}_{A_3}\}$ and $N_1 = \text{span}_K\{\partial_{B_1}, \partial_{B_2}, \partial_{B_3}, \partial_{B_4}, \partial_{D_5}, \partial_{D_6}\}$, then $N = N_0 \oplus N_1$ is a 9-dimensional Lie superalgebra of nilpotency class 4.
Central invariants

(see Table 1). Lemma 7 asserts that the subalgebra of invariants $R_{\tilde{L}}$ under the action of $\tilde{L}$ consists of all matrices of the form \[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \lambda
\end{pmatrix}
\] where $\alpha, \beta, \gamma, \lambda \in K$. Moreover,

\[
\partial_{D_3}\left(
\begin{pmatrix}
\alpha & \beta \\
\gamma & \lambda
\end{pmatrix}
\right) = \begin{pmatrix}
\gamma(\tilde{a}_3 - \tilde{a}_1) - \beta - \gamma & (\lambda - \alpha)(\tilde{b}_4 + \tilde{d}_5) \\
(\alpha - \lambda)\tilde{b}_4 & \gamma(\tilde{a}_3 + \tilde{a}_1) - \beta - \gamma
\end{pmatrix}.
\]

As a result we obtain that $R_{\tilde{L}} = K$.

Notice also that if $M_0 = \text{span}_K\{\text{ad} A_1, \text{ad} A_2, \text{ad} A_3\}$ and $M_1 = \text{span}_K\{\partial B_1, \partial B_2, \partial B_3, \partial B_4, \partial B_5 + B_7, \partial B_6\}$, then $M = M_0 \oplus M_1$ is a nilpotent Lie superalgebra of nilpotency class 6 (see Table 2). We have

\[
\partial_{B_5 + B_7}\left(
\begin{pmatrix}
\alpha & \beta \\
\gamma & \lambda
\end{pmatrix}
\right) = \begin{pmatrix}
\gamma(\tilde{a}_3 - \tilde{a}_1) - \beta - \gamma & (\lambda - \alpha)(\tilde{b}_4 + \tilde{d}_5) \\
(\alpha - \lambda)\tilde{b}_4 & \gamma(\tilde{a}_3 + \tilde{a}_1) - \beta - \gamma
\end{pmatrix}.
\]

This implies immediately that $R_M = K$.

Finally, observe also that $M$ is a subalgebra of a nilpotent Lie superalgebra $L = L_0 \oplus L_1$ of nilpotency class 6, where $L_0 = [L_1, L_1] = \text{span}_K\{\text{ad} A_1, \text{ad} A_2, \text{ad} A_3\}$ and $L_1 = \text{span}_K\{\partial B_1, \partial B_2, \partial B_3, \partial B_4, \partial B_5, \partial B_6, \partial B_7\}$ (see Table 2). Obviously, $R_{\tilde{L}} = K$. Starting with the algebra $R$, the invertible element $u = B_7$ and the Lie superalgebra $L$, and again applying the above procedure, we can produce successive examples.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\texttt{[\cdot,\cdot]} & \texttt{ad}_A_1 & \texttt{ad}_A_2 & \texttt{ad}_A_3 & \partial_B_1 & \partial_B_2 & \partial_B_3 & \partial_B_4 & \partial_B_5 \\
\hline
\texttt{ad}_A_1 & 0 & 0 & 0 & 0 & 0 & \texttt{ad}_A_1 & 0 & \partial_B_2 + \partial_B_3 \\
\texttt{ad}_A_2 & -2\partial_B_1 & 2\partial_B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\texttt{ad}_A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{operation table of $N$}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\texttt{[\cdot,\cdot]} & \texttt{ad}_A_1 & \texttt{ad}_A_2 & \texttt{ad}_A_3 & \partial_B_1 & \partial_B_2 & \partial_B_3 & \partial_B_4 & \partial_B_5 & \partial_B_6 \\
\hline
\texttt{ad}_A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\texttt{ad}_A_2 & 2\partial_B_1 & 2\partial_B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\texttt{ad}_A_3 & -2\partial_B_1 & -2\partial_B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_B_2 & 2\partial_B_1 & 2\partial_B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_B_3 & -2\partial_B_1 & -2\partial_B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_B_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{operation table of $L$}
\end{table}

References


