Gelfand-Kirillov dimension of factor algebras of Golod-Shafarevich algebras

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Inspiration

- 10 problems and 3 conjectures

  stated by Efim Zelmanov in the paper

  Some open problems in the theory of infinite dimensional algebras,

  J. Korean Math. Soc. 44 (2007), No. 5, 1185-1195.

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- We give more information on Problem 5 and Conjecture 3...

Outline

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- Problem 5 (Zelmanov, 2007)
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- Some recent results
- More open questions

Problem 5

PROBLEM 5 (Zelmanov, 2007)

Is it true that an arbitrary Golod-Shafarevich algebra has an infinite dimensional homomorphic image of finite Gelfand-Kirillov dimension?

Zelmanov's Problem 5

Theorem (A.S. 2008)

(Glasgow Mathematical Journal, to appear)

Let K be a field of infinite transcendence degree.

Then there is a Golod-Shafarevich algebra R such that every infinite-dimensional homomorphic image of R has exponential growth.

Something old, something new...

It is known that

Golod-Shafarevich algebras have exponential growth.

Theorem (A.S., 2008, Glasgow Mathematical Journal, to appear)

Non-nilpotent factor rings of generic Golod-Shafarevich algebras over fields of infinite transcendence degree have exponential growth, provided that the number of defining relations of degree less then n grows exponentially with n.

Golod-Shafarevich theorem

Let R_d be a noncommutative polynomial ring in d variables over a field K, and let I be the ideal generated by an infinite sequence of homogeneous elements of degree larger than 1, where the number of elements of degree i is equal to r_i.

If the coefficients of the power series

$$(1 - dt + \sum_{i=2}^{\infty} r_i \, t^i)^{-1}$$

are all nonnegative, then

the factor algebra R_d/I is infinite-dimensional.

Let

- $H(t) = \sum_{i=2}^{\infty} r_i t^i$.
- $A = R/I$.
- A is graded (each generator has degree 1), so

$$A(t) = \sum_{i=1}^{\infty} \dim_K A_i t^i.$$

- Golod and Shafarevich proved that

$$A(t)(1 - dt + H(t)) \geq 1.$$

It follows that if there is $t_0 > 0$ such that

$$H(t) = \sum_{i=2}^{\infty} r_i t^i$$

- converges at t_0
- and $1 - dt_0 + H(t_0) < 0$,

then $A = R/I$ is infinite dimensional.

Golod-Shafarevich algebra

We say that R_d/I is a Golod-Shafarevich algebra if there is a number $0 < t_0$ such that

$$H(t) = \sum_{i=2}^{\infty} r_i t^i$$

converges at t_0 and $1 - dt_0 + H(t_0) < 0$.

Great facts

Golod-Shafarevich algebras were used to solve

- the General Burnside problem,

It is known that Golod-Shafarevich algebras have exponential growth.

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- the Class Field Tower problem.

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- the General Burnside problem,
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Golod-Shafarevich groups

- **Every GS group is infinite.**

Golod-Shafarevich groups

- **Every GS group is infinite.**
- **Zelmanov (2000) showed**

that every GS -group contains

a nonabelian free pro-p group

as a subgroup.

Notation

- **Let K be a field, F- the prime subfield of K.**
- **Let R be a K-algebra.**
- **Given subsets S, Q of R, denote**

$$S + Q = \{s + q : s \in S, q \in Q\}$$

$$SQ = \{\sum_{i=1}^{n} s_i q_i : s_i \in s, q_i \in Q, n \in N\}$$

Notation

Given a subset S of $\mathbb{K}$.

- $\mathbb{F}[S]$ is the field extension of $\mathbb{F}$, generated by elements from S,
- FS is the linear space over $\mathbb{F}$ spanned by elements from S.
- $\mathrm{card}(S)$ is the cardinality of S.

Lemma

Lemma

Let K be a field, F be a prime subfield of K, let R be a K-algebra and M be a subset of R.

Let $N_1 = M$ and for each $i > 1$, let N_i be a subset of FM^i such that $KM^i = KN_i$.

Denote $\alpha_i = \mathrm{card}(N_i)$.

Then there are subsets $S_i \subseteq K$ such that

$$S_1 = \{1\},\ \mathrm{card}(S_{i+1}) \leq \mathrm{card}(S_i) + \alpha_{i+1}\alpha_i\alpha_1$$

and $M^i \subseteq F[S_i]N_i$ for all i.

Proof

Proof. We will proceed by induction on i.

For $i = 1$ it is true because $N_1 = M$.

Suppose the result holds for some i.

We will show it is true for $i + 1$. Observe that M^{i+1} consists of finite sums of elements $m_{i+1} = m_i m_1$ for some $m_i \in M^i$, $m_1 \in M$. By the inductive assumption $m_i \subseteq F[S_i]N_i$. Therefore, $m_{i+1} \subseteq F[S_i]N_iN_1$. Recall that $N_iN_1 \subseteq KM^{i+1} = KN_{i+1}$.

So...

Consequently, every element $n_i n_1$ with $n_i \in N_i$ and $n_1 \in N_1$ can be written as a linear combination over K of elements from N_{i+1}.

$$n_i n_1 = \sum_{n_{i+1} \in N_{i+1}} k_{n_{i+1}, n_i, n_1} n_{i+1}$$

for some $k_{n_{i+1}, n_i, n_1} \in K$.

Denote

$$K_{i+1} = \{k_{n_{i+1}, n_i, n_1} : n_{i+1} \in N_{i+1}, n_i \in N_i, n_1 \in N_1\}.$$

Observe that...

Observe that

$$N_i N_1 \subseteq F[K_{i+1}]N_{i+1}.$$

Denote
$S_{i+1} = S_i \cup K_{i+1}.$

Then,

$$M^{i+1} \subseteq F[S_i]N_i N_1 \subseteq F[S_{i+1}]N_{i+1}.$$

Note that $\mathrm{card}(S_{i+1}) \leq \mathrm{card}(S_i) + \mathrm{card}(K_{i+1})$, hence

$$\mathrm{card}(S_{i+1}) \leq \mathrm{card}(S_i) + \alpha_{i+1}\alpha_i\alpha_1.$$

Let K be a field and let F be the prime subfield of K.

We say that elements

$a_1, a_2, \ldots, a_n$ are algebraically independent over F

if the algebra generated over F

by elements $a_1, a_2, \ldots, a_n$ is free.

Main theorem (A.S.), 2008

Let K be a field, F- the prime subfield of K, let R be a K-algebra, M- a finite subset of R. Denote $\alpha_1 = \mathrm{card}(M)$ and for $i > 1$, $\alpha_i = \dim_K KM^i$. Let $m > 1, n, t$ be natural numbers and let $x_1, \ldots, x_t \in FM^m$. Assume that there are elements $k_{i,j} \in K$ which are algebraically independent over F and such that for all $i \leq n$ we have

$$\sum_{j=1}^{t} k_{i,j}\, x_j = 0.$$

If $n > 1 + \sum_{i=2}^{m} \alpha_i \alpha_{i-1} \alpha_1$, then

$$x_1 = x_2 = \ldots = x_t = 0.$$

Conjecture 3

Zelmanov's Conjecture 3, 2007

Let $A = A_1 + A_2 + \dots$ be a graded algebra generated by A_1, with $\dim A_1 = m$ and presented by less than $\frac{m^2}{4}$ generic quadratic relations.

Then all but finitely many Veronese subalgebras can be epimorpically mapped onto the polynomial ring $K[t]$.

Zelmanov's Conjecture 3, 2007

Theorem (A.S. 2008, Glasgow Mathematical Journal, to appear)

Let K be a field of infinite transcendence degree and let $m > 8$.

Then there exists a graded algebra $A = A_1 + A_2 + \dots$ generated by A_1, with $\dim_K A_1 = m$ and presented by less than $\frac{m^2}{4}$ quadratic relations such that, for every i, the subalgebra of A generated by A_i cannot be epimorpically mapped onto the polynomial ring $K[t]$.

Zelmanov's Conjecture 3, 2007

It is not known if in arbitrary quadratic Golod-Shafarevich algebras almost all Veronese subalgebras can be mapped onto algebras with linear growth, or onto a polynomial-identity algebras!

Some recent results

Jacobson radical

Theorem (A.S., Bull. London Math. Soc., 2008)
Let R be a ring, **s** be a subset of R, and let

$$P = \mathbf{s} + \mathbf{s}^2 + \dots$$

be a subring of R generated by **s**.

Suppose that all $n \times n$ matrices with coefficients from **s** are nilpotent for $n = 1, 2, \dots$.
Then

- for all natural numbers n, m, all $n \times n$ matrices with entries from $\mathbf{s}^m$ are nilpotent,
- ring P is Jacobson radical.

Jacobson radical

Theorem (A.S., Bull. London Math. Soc., 2008)

Over every field K, there is a graded algebra

$$R = \oplus_{i=1}^{\infty} R_i,$$

- **generated by 2 elements of degree 1,**
- **which has all homogeneous elements nilpotent**
- **and is not Jacobson radical.**

Around Regev's theorem

Theorem (Regev)

Associated graded graded algebras to algebraic algebras over uncountable fields are algebraic.

Theorem (A.S., 2008)

Associated graded algebras to nil algebras need not be algebraic.

Riley's question

Open question (Riley)

Are associated graded graded algebras to nil algebras Jacobson radical?

Theorem (A.S., 2008)

Associated graded algebras to

nil algebras need not be nil.

More open questions

Zelmanov's question

- I. Let **K** be a field of finite transcendence degree.

  Is it true that every Golod-Shafarevich algebra **K**-algebra has an infinite dimensional homomorphic image of finite Gelfand-Kirillov dimension?

Zelmanov's question

- I. Let **K** be a field of finite transcendence degree.

  Is it true that every Golod-Shafarevich algebra **K**-algebra has an infinite dimensional homomorphic image of finite Gelfand-Kirillov dimension?

- ?

Zelmanov's question

- II. Is it true that every finitely presented Golod-Shafarevich algebra has an infinite dimensional homomorphic image of finite Gelfand-Kirillov dimension?

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Zelmanov's question

- III. Is it true that in arbitrary Golod-Shafarevich algebras with all defining relations of degree 2 almost all Veronese subalgebras can be mapped

  * **onto algebras with linear growth,**

  or

  * * **onto a polynomial-identity algebras?**

Zelmanov's question

- III. Is it true that in arbitrary Golod-Shafarevich algebras with all defining relations of degree 2 almost all Veronese subalgebras can be mapped

  * **onto algebras with linear growth,**

  or

  * * **onto a polynomial-identity algebras?**

- ? ? ?

Golod-Shafarevich proved that if the series

$$\left(1 - d\,t + \sum_{i=2}^{\infty} r_i\,t^i\right)^{-1}$$

has all coefficients nonnegative, then

all free algebras in **d** generators subject to some relations $f_1, f_2, \ldots$ with r_i relations of degree i, are infinite dimensional.

Anick's question

- I. Suppose that the series

$$\left(1 - d\,t + \sum_{i=2}^{\infty} r_i\,t^i\right)^{-1}$$

has a negative coefficient.

Anick's question

- I. Suppose that the series

$$\left(1 - d\, t + \sum_{i=2}^{\infty} r_i\, t^i\right)^{-1}$$

  has a negative coefficient.

- Is there a finitely generated algebra in d generators subject to r_i relations of degree i for $i = 1, 2, \ldots$?

A quadratic Golod-Shafarevich algebra

is a free algebra in d generators subject to r relations of degree 2 with $4r < d^2$.

Then the series

$$(1 - d\ t + r\ t^2)^{-1}$$

has all coefficients nonnegative.

Such algebras are infinite dimensional.

Anick's question

- II. Let d be a number.

Anick's question

- II. Let d be a number.
- Is there a free algebra in d generators subject to

$$\frac{(1+d^2)}{4}$$

or less relations of degree 2 which is finitely dimensional?

Zelmanov's question

- IV. Is it true that every algebra in d generators subject to less than

$$\frac{d^2}{4}$$

relations of degree 2 can be mapped onto a matrix ring over a commutative ring?

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THANK YOU!

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