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TALKING ON RADICAL THEORY

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Sum of subrigs

Let R_1, R_2 be subrings of a ring R such that $R_1 + R_2 = R$, R_1 is nilpotent, and R_2 is nil.
Is R a nil ring?

Equivalently (Ferrero and Puczyłowski, 1989)

$$L_1, L_2 \triangleleft_\ell R, \quad L_1, L_2 \text{ nil} \implies L_1 + L_2 \text{ nil?}$$

Equivalently

$$L \triangleleft_\ell R, \text{ and } L \text{ nil} \implies L \subseteq \mathcal{N}(R)?$$

which is **Köthe's Problem** (1930)

Kegel, 1964

$$R_1, R_2 \subseteq R \text{ nilpotent} \quad \implies R_1 + R_2 \text{ nilpotent}$$

Kelarev, 1993

$$R_1, R_2 \text{ locally nilpotent} \ \text{ does not imply } R \text{ nil}$$

Beidar – Mikhalev, Kępczyk – Puczyłowski, Salwa, and others

Homomorphic images

Puczyłowski – Smoktunowicz, 1998

(PS) If R is nil, then $R[x]$ cannot be mapped onto a ring with unity

Beidar – Fong – Puczyłowski, 2001

(BFP) If R is nil, then $R[x]$ cannot be mapped onto a ring with $\neq 0$ idempotent

Krempa, 1972

(K) Can $R[x]$, with R nil, be mapped onto a primitive ring?

Equivalently

(K) Is $R[x]$ a Jacobson radical ring for all nil rings R?

This is equivalent to Köthe's Problem

(PS) is equivalent to

R is nil $\implies$ $R[x]$ is a Brown–McCoy radical ring

(BFP) is equivalent to

R is nil $\implies$ $R[x]$ is a Behrens radical ring

Beidar – Puczyłowski – Wiegandt, 2002

If R is nil, then $R[x]$ is in the upper radical class of uniformly strongly prime rings as well as in that of von Neumann regular rings.

Radicals of polynomial rings

Agata Smoktunowicz solved Amitsur's Problem in 2000:

A polynomial ring $R[x]$ over a nil ring R need not be nil

A (radical) class $\mathcal{C}$ of rings is said to be *polynomially extensible*, if $A \in \mathcal{C} \implies A[x] \in \mathcal{C} \quad \forall A$

Polynomially extensible radicals: Baer (prime) radical, Levitzki radical

Let γ, δ be radicals; δ is *polynomially extensible to* γ, if $A \in \delta \implies A[x] \in \gamma$

The nil radical $\mathcal{N}$ is polynomially extensible to the Behrens radical (Beidar, Fong and Puczyłowski, 2001)

Köthe's Problem: Is the nil radical $\mathcal{N}$ polynomially extensible to the Jacobson radical $\mathcal{J}$?

Tumurbat – Wiegandt, 2003

Open problems in

M. Ferrero, An Inroduction to Köthe's Conjecture and Polynomial rings, *Resenhas IME-USP*, 5 (2001), 139–149

E. R. Puczyłowski, Questions related to Köthe's Nil Ideal Problem, *Algebra & Appl., AMS Contemp. Math.* # 419 (2006), pp 269–283.

Let R be a nil ring and X a set of cardinality $|X| > 1$.

Is the polynomial ring $R[X]$ in commuting or noncommuting indeterminates a Brown-McCoy radical ring? (Ferrero and Wisbauer, 2003; Puczyłowski, 2006)

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Radicals of skew polynomial rings

Radicals of power series rings

Radicals of group rings and graded rings

Lattices of radicals

Which ideals can be radicals?

The Amitsur property

A radical γ has the *Amitsur property*, if

$$\gamma(A[x]) = (\gamma(A[x]) \cap A)[x] \quad \forall A$$

Krempa, 1972

A radical γ has the Amitsur property iff

$$\gamma(A[x]) \cap A = 0 \implies \gamma(A[x]) = 0 \quad \forall A$$

Examples: Baer (prime) radical, Levitzki radical, Köthe (nil) radical, Jacobson radical, Brown-McCoy radical

Loi – Wiegandt, 2006

A hereditary radical γ has the Amitsur property iff
(i) *the semisimple class $\mathcal{S}\gamma$ is polynomially extensible* ($A \in \mathcal{S}\gamma \implies A[x] \in \mathcal{S}\gamma$)
(ii) $f(x) \in \gamma(A[x]) \implies f(0) \in \gamma(A[x])$

Example: The generalized nil radical
$\mathcal{N}_g$ (the upper radical of all domains)
has the Amitsur property

Problem: Give an easly testable criterion for the Amistur property of a (hereditary) radical (which works also in the case of the Jacobson radical)

Radical and torsion theory of acts

S-act A over a monoid S:
$a \mapsto sa \in A \quad \forall a \in A, \quad s \in S$
$(st)a = s(ta) \quad \forall a \in A, \quad s, t \in S$
$1 \cdot a = a \quad \forall a \in A$

Wiegandt, 2006

Hereditary torsion theory (as for modules):

equivalence class $\mathcal{E}$ of injectives $\Leftrightarrow$ torsionfree class $\mathcal{F}$

torsionfree class $\mathcal{F}$ $\Leftrightarrow$ torsion assignment τ

τ is a hereditary Hoehnke radical, but not aways a Kurosh–Amitsur radical assignment

Therefore,

torsion assignment $\tau \implies$ torsion class $\mathcal{T}$

is not always one-to-one

For rings (and modules):
a hereditary Hoehnke radical is a Kurosh–Amitsur radical (torsion) assignment

Problem: Find a necessary and sufficient condition for an equivalence class $\mathcal{E}$ of injective S-acts (maybe, by redefining injectivity involving also congruences) such that the corresponding torsion assignment τ be a Kurosh–Amitsur radical

Passed away:
Amitsur, Andrunakievich, Beidar, Hoehnke, de la Rosa, Stewart, Suliński, Szász.

Over 70:
Anderson, Divinsky, Ferrero, Leavitt, van Leeuwen, Liu Shao-Xue, Ryabukhin, Sands, Shoji Kyuno, Weinert, Wiegandt, Xu Yong-Hua,

and some others over 60.

No wonder that in the recent years the development in radical theory has slowed down.

The reason is not the lack of interesting, important and difficult problems, but

that of a new generation,

as in many other branches of pure mathematics.