Radicals in classification of commutative reduced filial rings

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All rings in this talk are associative but we do not assume that each ring has an identity element. By $\mathbb{Z}$ we denote the ring of integers and by $\mathbb{N}$ the set of positive integers. Moreover, by $\mathbb{P}$ we denote the set of all prime integers.

An associative ring R is called **filial** if $A \triangleleft B \triangleleft R$ implies $A \triangleleft R$ for all subrings A, B of R.

Problem of describing filial rings was raised by Szász in [10]. The problem have been studied by various authors, namely, Andruszkiewicz [1], [2], [3], [4], Eherlich [5], Filipowicz, Puczyłowski [6], [7] Sands [9], Veldsman [11] and others.

In [2] the complete classification and the method of construction of commutative filial domains was given.

The classification was proceeded from considering the set

$$\Pi(R) = \{p \in \mathbb{P} : p \text{ is not a unit in } R\}.$$

It was shown that for an arbitrary subset Π of the set of primes, a ring R is a filial integral domain of characteristic 0 with $\Pi(R) = \Pi$ if and only if R is isomorphic to a subring of $\mathbb{Q}_\Pi = \prod_{p\in\Pi} \mathbb{Q}_p$ of the form $K \cap \mathbb{Z}_\Pi$, where $\mathbb{Z}_\Pi = \prod_{p\in\Pi} \mathbb{Z}_p$, K is a subfield of $\mathbb{Q}_\Pi$ such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$ we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and $\mathbb{Q}_p$ is the quotient field of the p-adic integers $\mathbb{Z}_p$.

A ring R is said to be **reduced** if it does not have nontrivial nilpotent elements. We say that R is a CRF-**ring** when R is a commutative reduced filial ring.

It follows from Andrunakievich-Ryabukhin Theorem that every reduced ring is a subdirect sum of domains. So, it seems that from the classification of filial integral domains we can obtain a structure theorem for CRF-rings. Observe that the ring $R = \mathbb{Z} \oplus \mathbb{Z}$ is not filial, though $\mathbb{Z}$ is filial and R is a subdirect sum of finite simple fields. Unfortunately, it is not possible to transfer theorems obtained for filial integral domains automaticaly to subdirect sums.

Notice that the class of filial rings is not closed under extensions.

A ring R is called **strongly regular** if for every $a \in R$, $a \in Ra^2$. Every strongly regular ring is reduced and the class $\mathbb{S}$ of all strongly regular rings is a radical class. It is easy to see that if $R \neq 0$ is a commutative domain and R is not a field then $\mathbb{S}(R) = 0$.

Theorem 1. *[Theorem 3.4, [6]] The following conditions on a ring R are equivalent:*
(i) R is reduced and left filial,
(ii) $R/\mathbb{S}(R)$ is a CRF-ring.

Above theorem gives a strong motivation to describe the class of $\mathbb{S}$-semisimple commutative reduced filial rings. (Reduced left filial rings are left duo i.e., left ideals of such rings are two-sided). Moreover we have following:

Proposition 2. *The class of all CRF-rings is equal to the class of all extensions of commutative strongly regular rings by $\mathbb{S}$-semisimple CRF-rings.*

Theorem 1 implies that every strongly regular ring is filial. Notice that the converse statement does not hold. (ex. $\mathbb{Z}$).
However, we have the following positive result:

$$R - CRF \Rightarrow R \in \mathbb{S}_a.$$

where $\mathbb{S}_a$ denotes the class of all almost strongly regular rings. Recall that a ring R is **almost strongly regular** if for every $x \in R$ there exists $n \in \mathbb{N}$ such that $nx \in Rx^2$. It is easy to see that $\mathbb{S}_a$ forms a radical class.

For every torsion-free ring R we denote by $\Pi(R)$ the set $\Pi(R) = \{p \in \mathbb{P} : pR \neq R\}$. If R has an identity then $\Pi(R) = \{p \in \mathbb{P} : p \notin R^*\}$ is an analogue to the set $\Pi(A)$ introduced in [2].

Let p be a prime number. We denote by $\mathcal{T}_p$ the class of all rings R such that $pR^+ = R^+$. Let us observe that $\mathcal{T}_p$ is a radical class. For every ring $R \in \mathcal{T}_p$ and for every $n \in \mathbb{N}$, $p^n R = R$. Moreover, if R is torsion-free then $\mathcal{T}_p(R) = \bigcap_{n=1}^{\infty} p^n R$.

Remark 3. *Let R be a torsion-free ring. For every prime p, $p \notin \Pi(R)$ if and only if $R \in \mathcal{T}_p$.*

In [3] some results concerning relations of the radical classes $\mathbb{S}_a$, $\mathcal{T}_p$ with filiality were obtained. For instance:

Proposition 4. *Let R be a torsion-free CRF-ring. Then for every prime p the ring $R/\mathcal{T}_p(R)$ is reduced.*

Theorem 5. *Let A and B are non-zero torsion-free CRF-rings such that $\mathcal{T}_p(A) = 0$ and $\mathcal{T}_p(B) = 0$ for some prime p. Then $A \oplus B$ is not filial.*

Theorem 6. *Let R be a non-zero torsion-free CRF-ring. Then:*
(i) $\exists_{p\in\mathbb{P}} : \mathcal{T}_p(R) = 0 \Rightarrow R$ *is a domain.*
(ii) $p \in \Pi(R) \Rightarrow \mathcal{T}_p(R) \lhd_1 R$.

Proposition 7. *Let R be a torsion-free commutative reduced ring. Then R is filial if and only if for every $x \in R$:*
(i) $Rx + \mathbb{Z}x = pRx + \mathbb{Z}x$ for every $p \in \mathbb{P}$ and
(ii) $R \in \mathbb{S}_a$.

Theorem 8. *Let R be a torsion-free commutative reduced ring such that $\Pi(R) \neq \emptyset$. Then R is filial if and only if:*
(i) $|R/pR| = p$ for every $p \in \Pi(R)$ and
(ii) $R \in \mathbb{S}_a$.

Corollary 9. *Let T be a non-empty subset of $\mathbb{N}$ such that for every $t \in T$ there exists a torsion-free CRF-ring R_t such that $\Pi(R_t) \neq \emptyset$. If for every distinct $t, s \in T$, $\Pi(R_t) \cap \Pi(R_s) = \emptyset$, then $R = \bigoplus_{t \in T} R_t$ is a torsion-free CRF-ring and $\Pi(R) = \bigcup_{t \in T} \Pi(R_t)$.*

Proposition 7 allows to prove following two important theorems:

Theorem 10. *[Theorem 2.1, [4]] Every torsion-free, CRF-ring is an essential ideal in some torsion-free, CRF-ring with an identity.*

Theorem 11. *[Theorem 4.4, [4]] Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Then a ring R is an $\mathbb{S}$-semisimple CRF-ring with identity, such that $\Pi(R) = \Pi$ if and only if R is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$.*

Theorem 12. *Given a ring R with an identity element, the following conditions are equivalent:*

(1) *R is a noetherian $\mathbb{S}$-semisimple CRF-ring.*

(2) *$R \cong \bigoplus_{i=1}^{n} D_i$, where D_i is a filial integral domain of characteristic* 0, *which is not a field for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_j) = \emptyset$ for $i \neq j$.*

Theorem 13. *The following conditions on a ring R are equivalent:*

(1) *R is a noetherian $\mathbb{S}$-semisimple CRF-ring.*

(2) *$R \cong \bigoplus_{i=1}^{n} m_i D_i$, where D_i is a filial integral domain of characteristic* 0, *which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_j) = \emptyset$ for $i \neq j$.*

Theorem 14. *The following conditions on a ring R are equivalent:*
(1) *R is a noetherian CRF-ring.*
(2) *$R \cong \left(\bigoplus_{j=1}^{k} F_j\right) \oplus \left(\bigoplus_{i=1}^{n} m_i D_i\right)$, where D_i is a filial integral domain of characteristic 0, which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and F_j is a field for every $j \in \{1, 2, \ldots, k\}$.*

Theorem 15. *The following conditions on a ring R are equivalent:*
(1) *R is a finitely generated CRF-rings.*
(2) *$R \cong \left(\bigoplus_{j=1}^{k} F_j\right) \oplus \left(\bigoplus_{i=1}^{n} m_i D_i\right)$ where D_i is a finitely generated subring of $\mathbb{Q}$ for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and F_j is a finite field for every $j \in \{1, 2, \ldots, k\}$.*

Let K be a subring of $\mathbb{Q}_\Pi$ with the same identity. Take any $a \in K$. Let us denote by $supp(a)$ the set $\{p \in \Pi : a_p \neq 0\}$. Then $\mathscr{B}_K = \{supp(a) : a \in K\}$ is a boolean algebra.

For every $Y \subseteq \Pi$ we define $\chi_Y = (a_p)_{p\in\Pi} \in \mathbb{Z}_\Pi$ to be:

$$a_p = \begin{cases} 0 & \text{if } p \notin Y \\ 1 & \text{if } p \in Y. \end{cases} \tag{1}$$

Lemma 16. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Let K be a subring of $\mathbb{Q}_\Pi$ with the same identity. Then K is a strongly regular ring if and only if for every $a \in K$ exists $b \in K$ such that $ab = \chi_{supp(a)}$. In particular if K is a strongly regular ring, then $\chi_Y \in K$ for every $Y \in \mathscr{B}_K$.*

Lemma 17. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Let K be a strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$. Put $S = K \cap \mathbb{Z}_\Pi$. Then:*

(1) *every ideal J of K is of the form $J = \{\frac{1}{n}i : i \in J \cap S, n \in \mathbb{N}\}$,*

(2) *if S is noetherian, then K is also noetherian,*

(3) *S contains a nonzero ideal which is a domain, if and only K contains a nonzero ideal which is a domain.*

Theorem 18. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Then R is an $\mathbb{S}$-semisimple CRF-ring with an identity without ideals which are domains, such that $\Pi(R) = \Pi$ if and only if R is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and boolean algebra $\mathscr{B}_K$ is atom-free.*

Theorem 19. *R is an $\mathbb{S}$-semisimple CRF-ring without ideals which are domains if and only if R is isomorphic to some essential ideal of a ring of the form $K \cap \mathbb{Z}_\Pi$ where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and boolean algebra $\mathscr{B}_K$ is atom-free.*

Example 20. *Let p be any prime number. Let $A_{i,k} = \{p^i t + k : t \in \mathbb{N}\}$ for $i \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, p^i - 1\}$. Let*

$$\mathfrak{D} = \left\{ \bigcup_{j=1}^{n} X_j : n \in \mathbb{N} \right\}.$$

It is easy to see that for $i_1 \leq i_2$

$$A_{i_1,k_1} \cap A_{i_2,k_2} = \begin{cases} A_{i_2,k_2} & \text{if } k_1 \equiv k_2 \mod p^{i_1} \\ \emptyset & \text{if } k_1 \not\equiv k_2 \mod p^{i_1}, \end{cases}$$

So every element of $\mathfrak{D}$ can be written as a disjont sum of sets $A_{i,k}$. It means that if $X, Y \in \mathfrak{D}$ then $X \cap Y \in \mathfrak{D}$. Next it is also clear that $A'_{i,k} = \mathbb{N} \setminus A_{i,k} = \bigcup_{j \in \{0,1,\ldots,p^i-1\}, j \neq k} A_{i,j} \in \mathfrak{D}$. So $\mathfrak{D}$ is a field of sets. Of course for every $A_{i,k}$, and for every $j > i$ we have $A_{i,k} \supsetneq A_{i,j}$.

Example 21 Let $\Pi = \{p_1, p_2, \dots\}$ be any infinite subset of prime numbers. Let $\mathfrak{D}$ be any attom-free boolean algebra of subsets of Π. In $\mathbb{Q}_\Pi$ we define

$$K = [a\chi_Y : Y \in \mathfrak{D}, 0 \neq a \in \mathbb{Q}]. \tag{2}$$

It is easy to see that

$$K = \langle a\chi_Y : Y \in \mathfrak{D}, 0 \neq a \in \mathbb{Q}\rangle. \tag{3}$$

Hence every nonzero $d \in K$ can be written in the form

$$d = a_1\chi_{Y_1} + a_2\chi_{Y_2} + \cdots + a_k\chi_{Y_k}. \tag{4}$$

where $0 \neq a_i \in \mathbb{Q}$, $\emptyset \neq Y_i \in \mathfrak{D}$ for every $i \in \{1, 2, \dots, k\}$ and $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and $supp(d) = Y_1 \cup Y_2 \cup \cdots \cup Y_k$.

We claim that K is strongly regular. Let d be as in (4). Put $d' = a_1^{-1}\chi_{Y_1} + a_2^{-1}\chi_{Y_2} + \cdots + a_k^{-1}\chi_{Y_k}$. Obviously $d' \in K$, moreover $d \cdot d' = \chi_{supp(d)} \in K$. So by Lemma 16 K is strongly regular subring of $\mathbb{Q}_{\Pi}$. Clearly $K \cap \mathbb{Z}_{\Pi} \neq \{0\}$. It is easy to see that $\mathscr{B}_K$ is atom-free, so Theorem 18 implies that $K \cap \mathbb{Z}_{\Pi}$ is a nonzero $\mathbb{S}$-semisimple CRF-ring, without an ideal which is a domain. Moreover $\Pi(K \cap \mathbb{Z}_{\Pi}) = \Pi$.

Question 1 Describe extensions of commutative strongly regular rings by $\mathbb{S}$-semisimple CRF-rings.

Question 2 Is every filial ring an ideal in some filial ring with an identity?

Question 3 Describe commutative filial rings.

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