.

These are notes on which the talk was based. They do not contain explanations and some additional information, which were presented during the talk.

THE CATEGORICAL APPROACH TO GENERAL RADICAL THEORY

A SURVEY

László Márki

Three (partly overlapping) periods:

1) Rings and 'similar' structures (1953–1982)
[+ Gardner (1989), group-based structures]

2) Very different other structures
(top. spaces, graphs, semilattice decomposition of semigroups)
[Hoehnke 1967, 1972] (1975–1988)

3) Rings and similar structures again (1994 —)

Amitsur (1954)

First paper: complete lattices
Second paper:
first restate main results in the language of ideals and homomorphisms of rings
then:

„The whole theory can be developed in a far wider class of mathematical objects. The largest field in which this can be done is that of the Lattice-ordered Bicategories of MacLane defined in 1950, which satisfy some additional axioms so that the two main isomorphisms hold and some minor properties of ideals in rings.
We do not intend to give here the list of axioms such a category has to satisfy, but only a list of conditions, some of which may be considered as axioms, others as lemmas, to be valid in a category in order that the whole theory can be developed in it. It is worth noting that the whole theory of radicals is just a relation between injections and projections of a bicategory. We just use 'objects', 'normal subobjects', 'supermaps' instead of 'rings', 'ideals' and 'homomorphisms'."

Šhulgeifer (1960)
↓ [Ryabukhin (1967)]
Andrunakievich – Ryabukhin (1979)

(AR1) $\mathcal{C}$ has a zero object

(AR2) every object possesses a representative set for its normal subobjects

(AR3) for each object its normal subobjects form a complete sublattice under the natural partial order of subobjects

(AR4) every morphism has a kernel

(AR5) every morphism has a normal epimorphic image

(AR6) normal epimorphisms carry normal subobjects into normal subobjects

Carreau (1971)

$\downarrow$

Holcombe – Walker (1978)

(HW1) $\mathcal{C}$ has a zero object, kernels and cokernels

(HW2) in $\mathcal{C}$ intersections of arbitrary sets of normal subobjects of an object exist, further, every object possesses a representative set for its normal subobjects

(HW3) every morphism factors through a cokernel followed by a monomorphism

(HW4) for every cokernel $t: X \twoheadrightarrow Y$ and every kernel $Z \rightarrowtail Y$ we have $t(t^{-1}(Z)) = Z$

$$\begin{tikzcd} & Z \arrow[d, tail] \\ X \arrow[r, two heads, "t"'] & Y \end{tikzcd}$$

(HW5) any set of normal subobjects of any object possesses a union which is again a normal subobject

---

Krempa and Terlikowska (1974)

Terlikowska-Osłowska (1977, 1978)

$=$

Dickson (1966)

"subfunctor of the identity"

Radical function: $\rho : C \rightsquigarrow \rho(C) \lhd C$

ρ-object: $\rho(C) = C$

Kurosh

(1) any normal epimorphic image of a ρ-object is itself a ρ-object

(2) if every non-zero normal epimorphic image of an object C has a non-zero normal subobject which is a ρ-object, then C is a ρ-object

Amitsur

(1)

(α) every object C has a unique maximal normal subobject which is a ρ-object

(iii) $\rho(C/\rho(C)) = 0$ for every object C

cokernel subcategory $\mathcal{E}(\mathcal{C})$

Carreau

radical functor $\rho : \mathcal{E}(\mathcal{C}) \longrightarrow \mathcal{C}$

(i) ρ is a subfunctor of the inclusion functor $\iota : \mathcal{E}(\mathcal{C}) \longrightarrow \mathcal{C}$

(ii) for every object C, $\rho(C)$ is a normal subobj. of C

(iii)

(complete) if B is a normal subobject of C and $\rho(B) = B$ then B is a subobject of $\rho(C)$

(idempotent) for every object C, $\rho(\rho(C)) = \rho(C)$

Hoehnke-radical (1967)

a radical function $\rho: C \rightsquigarrow \rho(C) \triangleleft C$

(iv) for every normal epimorphism $\varphi: C \twoheadrightarrow D$,
$\varphi(\rho(C))$ is a subobject of $\rho(\varphi(C))$

(iii) $\rho(C/\rho(C)) = 0$ for every object C

Radical functors (i) – (iii) are the same as Hoehnke-radicals.

A radical functor is complete and idempotent iff the corresponding Hoenke-radical satisfies
$(\rho(C) = 0$ and $B \triangleleft C) \Rightarrow \rho(B) \neq B$,
if $\rho(B) \neq B$ for all $B \triangleleft C$, then $\rho(C) = 0$.

Radical classes, semisimple classes,
radical-semisimple pairs

classes of objects $\mathbb{R}$, $\mathbb{S}$ in $\mathcal{C}$ such that

(I) $\mathbb{R} \cap \mathbb{S}$ consists of zero objects

(II) $A \in \mathbb{R}$, $A/B \neq 0 \Rightarrow A/B \notin \mathbb{S}$

(III) $A \in \mathbb{S}$, $B \triangleleft A \Rightarrow B \notin \mathbb{R}$

(IV) ~~for~~ every object A has a normal subobject B such that $B \in \mathbb{R}$ and $A/B \in \mathbb{S}$

(Mlitz)

Semisimple class:

(a) if $C \in \mathbb{S}$ and $B \overset{\text{non-zero}}{\triangleleft} C$ then B has a non-zero factorobject in $\mathbb{S}$

(b) $\mathbb{S}$ is closed under subdirect products

(c) for every object C put

$$(C)\mathbb{S} = \cap(B \triangleleft C : C/B \in \mathbb{S}),$$

then

$$(C(\mathbb{S}))\mathbb{S} = C(\mathbb{S})$$

---

semigroups, topological spaces, graphs, idempotent algebras, greatest semilattice image of a semigroup

$\longrightarrow$ TOTALLY NEW APPROACH

Márki – Mlitz – Wiegandt (1988)

- category with a unique factorization system
- trivial objects (⇐ Buys – Groenewald – Veldsman, 1981)
- radical is not a subobject but a factorobject (⇐ Hoehnke, 1967)
- radical is a functor related to the inclusion functor of the cokernel subcategory (⇐ Carreau)
- common generalization of Hoenke's M-relations (1967) and Gardner's (1984) systems of cosets of congruences
- technique not really categorical (⇐ Márki – Mlitz – Strecker, 1980)

---

constant morphism:
$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\eta_A"'] & & B \\ & TA \arrow[ur] & \end{tikzcd}$$

factors through the smallest factorobject of A

trivial object: η_A is an isomorphism

MMW axioms on the category

(I) $\mathcal{C}$ has a unique factorization system $(\mathcal{E},\mathcal{M})$

(II) the factorobjects of any object form a complete join semilattice

(III) the subobjects of any object form a meet semilattice

(IV) subobjects of trivial objects are trivial

(V) if (B_i) is an ascending chain of subobjects of any object C and γ is an ~~normal~~ epimorphism from $\mathcal{E}$ ~~from~~ $C \longrightarrow D$ such that the image of every B_i is a trivial subobject of D then the image of the join subobject $\vee B_i$ is also a trivial subobject of D

(VI) if $\gamma : C \longrightarrow D$ is an ~~normal~~ epimorphism from $\mathcal{E}$ then every maximal trivial subobject of D has an inverse image under γ

examples

M-relation, M-system

Σ – any system of subobjects of an object A

$\xrightarrow{A_\Sigma} A/\Sigma$ greatest factorobject of A such that the image of every subobject from Σ is trivial

Then each subobject from Σ is contained in the inverse image of a trivial subobject, hence also of a maximal trivial subobject of A/Σ.

$S(A)$ — the set of all systems Σ of non-trivial subobjects of A such that distinct subobjects from Σ are contained in the inverse images of distinct maximal trivial subobjects of A/Σ

M-relation:

$\Sigma M A$ where A is an object of $\mathcal{C}$, $\Sigma \in S(A)$, and

$\emptyset M A$ for all objects A,

$\{(A, 1_A)\} M A$ for every non-trivial object A,

if $\Sigma M A$ and D is a subobject of A then $\Sigma_D M D$ where $\Sigma_D = \{B \wedge D \mid B \in \Sigma\}$

meet as subobjects of A considered as a subobject of D

Σ – M-system

examples

M – radical:

(fix an M-relation in $\mathcal{C}$)

mapping $A \rightsquigarrow \xrightarrow{\rho_A} \rho(A)$
factorobject of A

such that

($\rho 1$) for every $\varphi : A \to C$ in $\mathcal{E}$
there is a $\rho\varphi : \rho(A) \to \rho(C)$ (in $\mathcal{E}$) with

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\rho_A"'] & C \arrow[d, "\rho_C"] \\ \rho(A) \arrow[r, "\rho\varphi"'] & \rho(C) \end{tikzcd}$$

commutative

($\rho 2$) for every object A, $\rho_{\rho(A)} = 1_{\rho(A)}$

($\rho 3$) $\rho_A = 1_A \iff$ in every non-empty $\Sigma M A$ there is a $B \in \Sigma$ with $\rho(B)$ non-trivial

($\rho 4$) for any $A \to C$ in $\mathcal{E}$ and $\Sigma M A$ such that $\rho(B)$ is trivial for every $B \in \Sigma$,
there exist an M-system Ξ in C with $\rho(D)$ trivial for all $D \in \Xi$
and a morphism $A/\Sigma \to C/\Xi$
such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "A_\Sigma"'] & C \arrow[d] \\ A/\Sigma \arrow[r] & C/\Xi \end{tikzcd}$$

is commutative

($\rho 1$) $\Rightarrow$ ($\rho 4$) if M is homomorphically closed

Radicals via pullback preservation (assoc. rings) (JM, 1994)

semisimple class:

(S) subdirectly closed

(H) hereditary

(E) closed under extensions

cat. $\mathcal{C} \rightsquigarrow \mathcal{C}_s$ all objects, surjective hom's

(S) $\Leftrightarrow$ (S_0) the inclusion functor $\mathbb{S}_s \to \mathcal{C}_s$ has a left adjoint I

$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[dr] & IA \arrow[d, dashed] \\ & X \end{tikzcd} \qquad (IA = A/\mathcal{R}A)$$

$\mathcal{C}$ has pullbacks:
$$\begin{tikzcd} B \times_Y X \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

~~($\mathbb{S}_2$) / A / X / E / B / Alleh~~

$\mathbb{S}$ is a semisimple class iff I preserves those pullbacks

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, two heads] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

in which g is surjective and $X, Y \in \mathbb{S}$.

OLD AXIOMS

(SA*1) *$(\mathcal{C})$ has a zero object, and any two objects have product and sum.*

(SA*2) *Any two monomorphisms have an intersection* (in other words: any two subobjects of any object A have an intersection).

(SA*3a) *Projections of products are normal epimorphisms.*

(SA*3b) (Image of normal epimorphisms) *For any normal epimorphism $p : E \to B$ and any monomorphism $w : F \to E$ there is a commutative diagram*

$$\begin{tikzcd} F \arrow[r, "q"] \arrow[d, "w"'] & C \arrow[d, "v"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

where v is a monomorphism and q is a normal epimorphism.

(SA*4) (Hofmann's axiom) *If in a commutative diagram of the above form* p, q *are normal epimorphisms,* v, w *are monomorphisms and* v *is normal, and if every normal monomorphism* $k : K \to E$ *with* $p \cdot k = 0$ *factors through* w*, then* w *is also normal.*

$$\begin{tikzcd} & F \arrow[r, "q"] \arrow[d, "w"'] & C \arrow[d, "v"] \\ K \arrow[ur, dashed, "l"] \arrow[r, "k"] & E \arrow[r, "p"] & B \end{tikzcd}$$

(SA*5) (Inverse image of normal epimorphisms) *For any normal epimorphism* $p : E \to B$ *and every monomorphism* $v : C \to B$ *there is a commutative diagram of the above form in which* w *is a monomorphim and* q *is a normal epimorphism.*

(SA*6) (The image of a normal monomorphism is normal) *If in a commutative diagram as above,* p, q *are normal epimorphisms,* v, w *are monomorphisms and* w *is normal, then* v *is also normal.*

NEW AXIOMS

(SA1) *($\mathcal{C}$) has a zero object, and any two objects have product and sum.*

(SA2) *Pullbacks of (splitting) monomorphisms exist.*

(SA3) *Every kernel pair has a coequaliser.*

(SA4) *The Splitting Short Five Lemma holds:*

$$\begin{tikzcd}
L \arrow[r, "\ker q"] \arrow[d, "u"'] & F \arrow[r, shift left, "q"] \arrow[d, "w"'] & C \arrow[l, shift left, dashed] \arrow[d, "v"] \\
K \arrow[r, "\ker p"] & E \arrow[r, shift left, "p"] & B \arrow[l, shift left, dashed]
\end{tikzcd}$$

if p, q are splitting epimorphisms, u, v are isomorphisms, then w is also an isomorphism.

(SA5) *Regular epimorphisms are preserved by pullbacks.*

(SA6) *Equivalence relations are effective.*

Axioms (SA1-3), (SA5-6) could be replaced here by the following:

(P) *(C) has a zero object, and any two objects have a sum.*

(Ex1) *Finite limits exist.*

(Ex2) *(Regular epi, mono) factorizations exist and are preserved by pullbacks.*

(Ex3) *Equivalence relations are effective.*

(Ex1-3) is just the definition of Barr exact categories.

Theorem. *The old and the new systems of axioms are equivalent.*

Remark. The old system of axioms gives a (normal epi, mono) factorization, and an essential part of the proof in one direction consists just in showing that regular epimorphisms are normal under the new system of axioms.

Remark. The old system of axioms satisfies the requirements of modern category theory far less than the new system does; on the other hand, it is easier to apply just because (partly in view of Orzech's results) normal epimorphisms are easier to handle than regular ones.

EXAMPLES OF SEMI-ABELIAN CATEGORIES

1. *Abelian categories.*

2. *Varieties of Ω-groups.* E.g.: groups, rings, Lie algebras, Jordan algebras, lattice ordered groups, etc.

3. *Internal varieties.* Let $\mathcal{T}$ be a semi-abelian variety and $\mathcal{C}$ be a Barr exact category. The category of $\mathcal{T}$-algebras in $\mathcal{C}$ is semi-abelian if and only if it admits finite sums.

4. *Heyting algebras.*

5. *The dual of the category of sets with a base point.*

Radicals via Galois connection

Torsion theories (Dickson, 1965)

$R \perp S \iff \mathrm{Hom}(R,S) = 0$

Assoc. rings, radical-semisimple pairs (Gardner, 1974)

$R \perp S \iff$ there is no non-zero homomorphism $f: R \longrightarrow S$ with $f(R)$ accessible in S

═

(Fried–Wiegandt, 1975)

Consider two preorders $\mapsto$ and $\prec$ in a class $\mathcal{A}$ of objects.

C-class $\mathcal{C}$:

$A \in \mathcal{C} \iff \forall B \in \mathcal{A} \quad A \mapsto B \Rightarrow \exists C \in \mathcal{C} \quad C \prec B.$

D-class $\mathcal{D}$:

$A \in \mathcal{D} \iff \forall B \in \mathcal{A} \quad B \prec A \Rightarrow \exists C \in \mathcal{D} \quad B \mapsto C.$

═

$\mathbb{R} = \{ R \in \mathcal{C} \mid R \perp S \text{ for all } S \in \mathbb{S} \}$

$\mathbb{S} = \{ S \in \mathcal{C} \mid R \perp S \text{ for all } R \in \mathbb{R} \}$

═

Binary relation α on a set X

$\Rightarrow$ Galois connection of the power set $P(X)$ with itself,

$$\alpha^*: P(X) \to P(X), \quad \alpha^*(U) = \{x \in X \mid u \in U \Rightarrow u \alpha x\}$$

Take two binary relations α, β on X, consider

$$P(X) \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} P(X);$$

largest induced bijection

$$\{U \subseteq X \mid U = \beta^* \alpha^*(U)\} \leftrightarrow \{V \subseteq X \mid V = \alpha^* \beta^*(V)\}$$

between left-closed and right-closed subsets.

U is left closed:

(a) if $x \in U$ then $(\forall u \;\; u \in U \Rightarrow u \alpha v) \Rightarrow v \beta x$

(b) if $(\forall u \;\; u \in U \Rightarrow u \alpha v) \Rightarrow v \beta x$ then $x \in U$

right-closed sets dually

Example for rings

$K \alpha A \Leftrightarrow K$ is not a non-zero ideal of A

$B \beta A \Leftrightarrow B$ is not a non-zero factor ring of A

1-dimensional exactness structure:

$$X = (X, 0, \triangleleft, \to)$$

with X a pointed set, $\triangleleft$ and $\to$ binary relations on X with $0 \triangleleft x$ and $0 \to x$ for all x in X.

Radical and semisimple class:

left- resp. right-closed classes for the relations

$$\alpha = \{(x,y) \mid x \triangleleft y \Rightarrow x = 0\}, \quad \beta = \{(x,y) \mid y \to x \Rightarrow x = 0\}$$

Consider

$$K \underset{f'}{\overset{f}{\leftrightarrows}} L \underset{g'}{\overset{g}{\rightleftarrows}} K'.$$

This gives a largest induced pair of bijections

$$\overline{K} \leftrightarrow \overline{L} \leftrightarrow \overline{K}' \qquad (*)$$

with

$$\overline{L} = \{\ell \in L \mid f'f(\ell) = \ell = g'g(\ell)\}$$
$$\overline{K} = f(\overline{L})$$
$$\overline{K}' = g(\overline{L}).$$

We have in mind

$K = K' =$ all classes of rings

$L =$ all functions which associate to any ring an ideal of it

$f(\ell) =$ all rings of the form $\ell(A)$

$f'(X)$ for $X \in K$: the function

$f'(X)(A) =$ the join of all ideals of A which belong to X

$g(\ell) =$ all rings of the form $A/\ell(A)$

$g'(X)$ for $X \in K'$: the function

$g'(X)(A) =$ the intersection of all ideals I of A for which A/I is in X

Then the bijections $(*)$ are precisely the canonical bijections between the radical classes, the radical functions, and the semisimple classes.

2-dimensional exactness structure

We want to model:

$\mathcal{C}$ — category of rings

X_1 — set of objects of $\mathcal{C}$

X_2 — equivalence classes of short exact sequences

$P/\sim$ where $(A \to B \to C) \sim (A' \to B' \to C')$

iff

$B = B'$ and
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{iso}"'] & B \arrow[r] \arrow[d, equal] & C \arrow[d, "\mathrm{iso}"] \\ A' \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

maps $e_0 : X_1 \longrightarrow X_2 : A \longmapsto (0 \longrightarrow A = A)$

$e_1 : X_1 \longrightarrow X_2 : A \longmapsto (A = A \longrightarrow 0)$

$d_0 : X_2 \longrightarrow X_1 : (A \to B \longrightarrow C) \longmapsto A$

$d_1 : X_2 \longrightarrow X_1 : (A \longrightarrow B \longrightarrow C) \longmapsto B$

$d_2 : X_2 \longrightarrow X_1 : (A \longrightarrow B \longrightarrow C) \longmapsto C$

$L = \{ \ell : X_1 \longrightarrow X_2 \mid d_1 \ell = 1_{X_1} \} = \prod_{x \in X_1} L_x$,

$L_x = \{ u \in X_2 \mid d_1 u = x \}$ is a complete lattice with smallest and largest elements $e_0(x)$ and $e_1(x)$

Consider

(a) a diagram in the category of sets

$$\begin{array}{c} X_2 \\ \downarrow d_0 \ \uparrow e_0 \ \downarrow d_1 \ \uparrow e_1 \ \downarrow d_2 \\ X_1 \end{array}$$

with

$$d_0 e_1 = d_1 e_0 = d_1 e_1 = d_2 e_0 = 1_{X_1}$$

(b) an element $0 \in X_1$ with
$e_0(0) = e_1(0)$ and $d_0 e_0(x) = d_2 e_1(x) = 0 \quad \forall x \in X_1$

(c) a complete lattice structure on each set
$$L_x = \{u \in X_2 \mid d_1(u) = x\} \qquad (x \in X_1) \ .$$
such that $e_0(x)$ and $e_1(x)$ are the smallest and the largest elements in L_x

Consider the diagram

$$K \underset{\Sigma_f}{\overset{f}{\leftrightarrows}} L \underset{\Pi_g}{\overset{g}{\rightleftarrows}} K$$

where

$$L = \{l : X_1 \longrightarrow X_2 \mid d_1 l = 1_{X_1}\} = \prod_{x \in X_1} L_x$$

K = complete lattice of all subsets of X_1 which contain 0

$$f(l) = d_0 l(X_1), \quad g(l) = d_2 l(X_1)$$

$$\Sigma_f(k) = \bigvee_{f(l) \leq k} l \ , \quad \Pi_g(k) = \bigwedge_{g(l) \leq k} l$$

A map $r \in L$ is called a <u>radical function</u> if

$$\sum_f (f(r)) = r = \prod_g (g(r)).$$

$R \subseteq X_1$ is a <u>radical class</u> if
there exists a radical function r with $f(r) = R$
(and then $r = \Sigma_f(R)$)

$S \subseteq X_1$ is a <u>semisimple class</u> if
there exists a radical function r with $g(r) = S$
(and then $r = \Pi_g(S)$).

Possible levels of interpretation: $\mathcal{C}$ is a

- category with zero object, kernels and cokernels
- semi-abelian category such that, for every object A, the ordered set $NSub(A)$ is a complete lattice
- semi-abelian variety
- a variety of multioperator groups
- the variety of all (not necessarily associative) rings
- the variety of associative rings