Radicals and classes of filial algebras

Marzena Filipowicz-Chomko

mfilipowicz@kki.net.pl

Bialystok Technical University, Poland

Workshop **Radicals of rings and related topics**
Warsaw, Poland

Outline

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1. Introduction
2. Semiprime algebras
3. Prime radical
4. The structure of left filial algebras over field F
5. Tzintzis radical

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Notation

- A - an associative algebra over a commutative ring K with identity
- $A^{\star}$ - the algebra A with a unity adjoined
- $\beta(A)$ - the prime radical of A
- $\mathcal{S}(A)$ - the strongly regular radical of A
- $\mathbf{Z}$ - the ring of integers
- $\mathbf{Z}_n$ - the factor ring $\mathbf{Z}/n\mathbf{Z}$
- $I \lhd A$ ($I <_l A$, $I <_r A$) - I is a K-ideal (a left K-ideal, a right K-ideal) of A
- $l_A(X) = \{a \in A \mid aX = 0\}$- left anihilator of X in algebra A.

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Fililal rings are

- analogous to t-groups (i.e. groups in which the relation of being a normal subgroup is transitive);
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Examples

Examples (of filial rings)

(1) H-rings (Hamiltonian rings), for instance $\mathbf{Z}$, $\mathbf{Z}_n$

(2) Simple rings

(3) Von Neuman regular rings

(4) Subidempotent rings ($\{R \mid \forall_{I \triangleleft R} I = I^2\}$)
More generally, the rings $\{R \mid \forall_{I \triangleleft R} RI = I^3 = IR\}$

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Semiprime algebras over an arbitrary commutative ring K with identity

We get the following results

- Every semiprime left filial algebra is reduced.
- A reduced algebra A is left filial if and only if for every $a \in A$ $aA^{\star} = aA^{\star}a + Ka$.
- A prime algebra is left filial if and only if it is a commutative filial domain or a division algebra.

From the above results we obtain the following structure theorem describing left filial semiprime algebras.

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Recall that an algebra A is called strongly regular if for every $a \in A$ there is $x \in A$ such that $a = a^2x$ (equivalently, $a = xa^2$).

Theorem

The following conditions on A are equivalent:

(i) *A is semiprime and left filial.*

(ii) *A contains an ideal I such that I is strongly regular and A/I is a commutative reduced filial algebra.*

(iii) *$A/S(A)$ is a commutative reduced filial algebra.*

Corollary. Every semiprime left filial algebra is filial.

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A β-radical algebra A is left filial if and only if $A^2 = 0$ or $A = B \oplus C$, where $C^2 = 0$, $B^2 = Fb$ for some $0 \neq b \in B$ with $Bb = bB = 0$ and for every $x \in B \setminus Fb$, $x^2 \neq 0$.

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The structure of left filial algebras over field F

Theorem

A is a left filial algebra if and only if $A/\beta(A)$ is strongly regular, $\beta(A)$ is H-algebra and

(i) $A = l_A(\beta(A)) + \beta(A)$

or

(ii) $A = Fe + l_A(\beta(A)) + \beta(A)$, *where* $\beta(A) \neq 0$ *and* e *is an idempotent of* A *such that* $eb = b$ *for every* $b \in \beta(A)$.

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Let U be an algebra with identity, T an algebra and M a $U - T$-bimodule, which is unitary as the left U-module. The set $\begin{pmatrix} U & M \\ 0 & T \end{pmatrix}$ of matrices of the form $\begin{pmatrix} u & m \\ 0 & t \end{pmatrix}$, where $u \in U$, $m \in M$ and $t \in T$, is an algebra with respect to the obvious matrix operations.

Theorem

A is an algebra satisfying case (ii) if and only if $A \simeq \begin{pmatrix} S^\star & M \\ 0 & T \end{pmatrix}$, where S is a left filial algebra such that $S = l_S(\beta(S)) + \beta(S)$ and $\beta(S) \neq 0$, T is a strongly regular algebra, M is an $S^\star - T$-bimodule, which is unitary as the left $S^\star$-module and such that $SM = 0$.

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Tzintzis got a satisfactory description of $\mathcal{X}_r$ and $\mathcal{X}_l$. Namely he obtained

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$\mathcal{X}_l = \mathcal{X}_r = u_{\mathcal{D}\cup\mathcal{T}}$, where $u_{\mathcal{D}\cup\mathcal{T}}$ is the upper radical determined by the union of the class of division rings $\mathcal{D}$ and the class of rings with zero multiplication $\mathcal{T}$.

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In above paper was constructed a ring giving a counterexample to all of Tzintzis' questions.

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