Accessible subrings and Kurosh's chains of associative rings

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Let $\mathcal{N}$ be a homomorphically closed class of associative rings. Put $\mathcal{N}_1 = \mathcal{N}^1 = \mathcal{N}$ and for ordinals $\alpha \geq 2$, define $\mathcal{N}_\alpha$, ($\mathcal{N}^\alpha$) to be the class of all associative rings R such that every non-zero homomorphic image of R contains a non-zero ideal in $\mathcal{N}_\beta$ (left ideal in $\mathcal{N}^\beta$) for some $\beta < \alpha$. In this way we obtain a chain $\{\mathcal{N}_\alpha\}$ ($\{\mathcal{N}^\alpha\}$), the union of which is equal to the lower radical class $l\mathcal{N}$ (lower left strong radical class $ls\mathcal{N}$) determined by $\mathcal{N}$. The chain $\{\mathcal{N}_\alpha\}$ is called *Kurosh's chain* of $\mathcal{N}$.

In [18] Suliński, Anderson and Divinsky studied the Kurosh's chain in the universal class of associative rings. They described the classes $\mathcal{N}_\alpha$ in terms of accessible subrings and proved that the chain stabilizes at the first limit ordinal ω. They asked whether, for each ordinal $\alpha \leq \omega$, there exists a class $\mathcal{N}$ such that $l\mathcal{N} = \mathcal{N}_\alpha \neq \mathcal{N}_\beta$ for $\beta < \alpha$. The question turned out to be very interesting and challenging. In [11], Heinicke answered it in the positive for $\alpha = \omega$. However, the problem for positive integers resisted effords of many authors for a long time. Various studies, except some general results, effected in solving the problem for small integers and determining when the chain stabilizes for some specific classes $\mathcal{N}$ ([17]).

The general problem was finally solved in the positive by Beidar in [5]. After that, several new examples were found (see [1], [2], [6], [13], [14], [19]). All of them were not easy to handle and required quite complicated arguments. Developing some ideas of the mentioned papers (mainly those of [14]), we construct new general examples. The generality allows us to avoid particular calculations which, we hope, makes the arguments clearer and more visible. It also allows us to construct radicals which satisfy some extra properties.

To denote that I is an ideal (left ideal) of a ring R, we write $I \lhd R$ $(I < R)$. Let n be a positive integer. A subring A of a ring R is said to be *n-accessible (left n-accessible)* in R if there are subrings $R = A_0, A_1, \ldots, A_{n-1}, A_n = A$ of R such that $A_i \lhd A_{i-1}$ $(A_i < A_{i-1})$ for $i = 1, 2, \ldots, n$, and A is said to be *precisely n-accessible* if it is not k-accessible in R for any positive integer $k < n$. A subring A is *accessible* (*left accesible*) in R if it is n-accessible (left n-accessible) for certain $n \in \mathbb{N}$.

The following proposition collects some well known properties of classes $\mathcal{N}_\alpha$ and $\mathcal{N}^\alpha$.

Proposition 1. (i) $\mathcal{N}_\alpha \subseteq \mathcal{N}^\alpha$ for every ordinal α;

(ii) classes $\mathcal{N}_\alpha$ and $\mathcal{N}^\alpha$ are homomorphically closed for all α;

(iii) if $0 \neq R \in ls\mathcal{N}$ then R contains a non-zero left accessible subring in $\mathcal{N}$;

(iv) $R \in l\mathcal{N}$ if and only if every non-zero homomorphic image of R contains a non-zero accessible subring in $\mathcal{N}$;

(v) if $0 \neq R \in \mathcal{N}^{n+1}$, where n is an integer ≥ 1, then R contains a non-zero n-left accessible subring in $\mathcal{N}$;

(vi) $R \in \mathcal{N}_{n+1}$, where n is an integer ≥ 1, if and only if every non-zero homomorphic image of R contains a non-zero n-accessible subring in $\mathcal{N}$;

(vii) ([8,16]) For every ordinal $\alpha \geq \omega$ and for every class $\mathcal{N}$, $\mathcal{N}^\alpha$ is a radical.

Recall that a radical class $\mathcal{S}$ is called *left stable* if for every $L < R$, $\mathcal{S}(L) \subseteq \mathcal{S}(R)$. An example of a left stable radical class is the generalized nil radical $\mathcal{N}_g$; this is the upper radical determined by the class of domains. For every homomorphically closed class $\mathcal{N}$ there exists the smallest left stable radical $st\mathcal{N}$ containing $\mathcal{N}$. Moreover, $R \in st\mathcal{N}$ if and only if every non-zero homomorphic image of R contains a non-zero left accessible subring in $\mathcal{N}$. For example:
$\mathcal{N}_g = st\{R \mid R^2 = 0\}$.

Theorem 1 ([1,15]). If $\mathcal{S}$ is a left stable radical class containing $\mathcal{N}_g$ then for $\mathcal{N} = \mathcal{S} \cup \mathcal{P}$, where $\mathcal{P}$ is a homomorphically closed class of commutative rings, $\mathcal{N}_\alpha = \mathcal{N}^\alpha$ for every ordinal α and $l\mathcal{N}$ is left stable. Moreover, if $\mathcal{N}$ is hereditary then $l\mathcal{N}$ is hereditary.

Note that if A is a subring of a commutative ring R, then for every positive integer n, $A + RA^{n+1}$ is the ideal of $A + RA^n$ generated by A. This gives us the following.

Proposition 2. Let A be a subring of a commutative ring R. Then
(i) A is n-accessible in R if and only if $RA^n \subseteq A$;
(ii) $A + RA^n$ is the smallest n-accessible subring in R containing A.

Given an element a in a commutative ring R, we shall denote by $\langle a \rangle$ the subring of R generated by a.

Observe that for every positive integer n, $\langle a \rangle + Ra^n = \langle a \rangle + R\langle a \rangle^n$. Hence, Proposition 2 implies that $\langle a \rangle + Ra^n$ is an n-accessible subring of R and for $n \geq 2$, the subring is precisely n-accessible in R if and only if $\langle a \rangle + Ra^n \neq \langle a \rangle + Ra^{n-1}$.

If R is a commutative ring and $a \in R$ is such that for every integer $n \geq 2$, $\langle a \rangle + Ra^n \neq \langle a \rangle + Ra^{n-1}$, then we write $acc_R(a) = \infty$.

Proposition 3 ([2]). Suppose that R is an integral domain and $0 \neq a \in R$. If $acc_R(a) = \infty$, then $R \neq \langle 1 \rangle + Ra$.
Conversely, if $R \neq \langle 1 \rangle + Ra$, then $acc_R(a) = \infty$ provided $Ra \cap \langle 1 \rangle = 0$. If $R \neq \langle 1 \rangle + Ra$ and $Ra \cap \langle 1 \rangle \neq 0$, then $Ra \cap \langle 1 \rangle = b\langle 1 \rangle$ for some $b \in \langle 1 \rangle$ and $acc_R(b) = \infty$.

A ring R is called *filial* if every accessible subring of R is an ideal of R. In [9], it was proved that an integral domain R is filial if and only if for every $0 \neq a \in R$, $R = \langle 1 \rangle + Ra$. Hence, by Proposition 3, an integral domain R is non-filial if and only if R contains an element a with $acc_R(a) = \infty$.

Applying Proposition 3, one can easily find integral domains R and elements $a \in R$ with $acc_R(a) = \infty$.

The following are particular examples.

1. Let A be an integral domain and let $R = A[x]$ be the ring of polynomials over A in the indeterminate x. Clearly, if $f \in R$ and $\deg f \geq 2$, then by Proposition 3, $acc_R(f) = \infty$.

2. Let A be an integral domain and $R = A[[x]]$ be the power series ring over A in the indeterminate x. For every integer $k \geq 2$, $acc_R(x^k) = \infty$. Moreover, if $A \neq \langle 1 \rangle$, then $acc_R(x) = \infty$.

3. Let R be an integral domain which is an algebra over a field F. If $0 \neq a \in R$ and $\dim_F(R/Ra) > 1$, then by Proposition 3, $acc_R(a) = \infty$. In particular, if $0 \neq r \in R$ and r is a non-invertible element of R, then $acc_R(r^2) = \infty$.

We say that a subring A of a ring R is *essential* in R if for every non-zero ideal I of R, we have $A \cap I \neq 0$.

Definition 1 (Beidar, 1993). We say that a ring R is a *iterated maximal essential extension* of a ring A and we write $R = IME(A)$ if A is an essential accessible subring of R and, for every ring S in which A is accessible, there exists a homomorphism of S into R which is the identity map on A.

Proposition 4 ([3]). Assume A is a semiprime ring for which there exists $R = IME(A)$. If A is n-accessible in R, then for every semiprime ring S in which A is accessible, A is n-accessible in S.

Proposition 5 ([3]). Let R be a ring such that $R = IME(I)$ for every non-zero ideal I of R. Let A and B be non-zero accessible isomorphic subrings of R. If A is precisely n-accessible in R, then so is B.

Let P be an integral domain with K the field of fractions. We say that P is a *completely normal ring* if for any $x \in K$ and $0 \neq a \in P$, we have that $\langle x \rangle a \subseteq P$ implies $x \in P$.
The following are particular examples:
- noetherian integrally closed domains,
- unique factorization domains,
- Krull rings.

Proposition 6 ([3]). If $R \neq 0$ is a commutative ring then the following conditions are equivalent:
(i) $R = IME(I)$ for every $0 \neq I \lhd R$,
(ii) $R = IME(A)$ for every non-zero accessible subring A of R;
(iii) R is a completely normal ring.

Proposition 7. Let R be a ring such that $R = IME(I)$ for every non-zero ideal I of R. Then the following conditions are equivalent:
(i) If $f\colon A \to B$ is an epimorphism of non-zero accessible subrings of R, then f is an isomorphism;
(ii) If $0 \neq I \lhd R$ and $g\colon R \to R$ is a homomorphism such that $Ker\ g \neq 0$, then $I \not\subseteq g(R)$.

Examples of rings R satisfies the condition (i) of Proposition 7:

- integral domains P such that P^+ is torsion-free and $(P/I)^+$ is torsion for every $0 \neq I \lhd P$ (cf. [5]),
- commutative domains A such that A/I is nilpotent for every $0 \neq I \lhd A$ (cf. [19]),
- noetherian integrally closed domains (cf. [2]),
- integral domains P with $K \neq P$ the field of fractions of positive characteristic p such that $\deg tr_F(P) = 1$ ($F = \mathbb{Z}/(p)$) (cf. [14]).

Theorem 2. Suppose that P is a non-filial completely normal ring such that if $f\colon A \to B$ is an epimorphism of non-zero accessible subrings of P, then f is an isomorphism; $\mathcal{P}$ is the class of all proper homomorphic images of all accessible subrings of P and $\mathcal{S} = st(\mathcal{P} \cup \{R | R^2 = 0\})$. Then for every $n \in \mathbb{N}$ the set $\mathcal{P}(n)$ of all precisely n-accessible subrings of P is non-empty. Moreover, if $\emptyset \neq \mathcal{A}(n) \subseteq \mathcal{P}(n)$ and $\mathcal{N}(n) = \mathcal{S} \cup \{\mathcal{A}(n)\}$, then $\mathcal{N}(n)_n = \mathcal{N}(n)^n \neq \mathcal{N}(n)_{n+1} = \mathcal{N}(n)^{n+1} = l\mathcal{N}(n) = ls\mathcal{N}(n) = st\mathcal{N}(n)$. If $\mathcal{A}(n) = \mathcal{P}(n)$, then the radical class $l\mathcal{N}(n)$ is hereditary.

Proof. Applying Proposition 3, we get that $\mathcal{P}(n) \neq \emptyset$ for every $n \in \mathbb{N}$. By Proposition 6, $P = IME(A)$ for every non-zero accessible subring A of P. Moreover, by Theorem 1, $\mathcal{N}(n)_\alpha = \mathcal{N}(n)^\alpha$ for every ordinal α and $l\mathcal{N}(n)$ is left stable.

Clearly, every non-zero homomorphic image of P contains a non-zero accessible subring in $\mathcal{N}(n)$. Hence, $P \in l\mathcal{N}(n)$. By the assumptions, $\mathcal{S}(P) = 0$ and by Proposition 5, no non-zero $(n-1)$-accessible subrings of P is in $\{\mathcal{A}(n)\}$. Consequently, $P \in l\mathcal{N}(n) \setminus \mathcal{N}(n)_n$.

We show that $l\mathcal{N}(n) = \mathcal{N}(n)_{n+1}$ or, equivalently, that every non-zero ring $R \in l\mathcal{N}(n)$ contains a non-zero n-accessible subring in $\mathcal{N}(n)$. This is obvious if $\mathcal{S}(R) \neq 0$. Thus, assume that $\mathcal{S}(R) = 0$, which implies that R is semiprime. Since $R \in l\mathcal{N}(n)$, R contains a non-zero accessible subring $D \in \mathcal{N}(n)$. Obviously, $D \notin \mathcal{S}$. Therefore $D \in \{\mathcal{A}(n)\}$, and by Proposition 4, D is a n-accessible subring of R.

Finally, suppose $\mathcal{A}(n) = \mathcal{P}(n)$. Let $\mathcal{A}$ be the class of all homomorphic images of all accessible subrings of P. Then by Theorem 1 the radical class $\mathcal{T} = st(\mathcal{A} \cup \{R | R^2 = 0\})$ is hereditary. We show that $l\mathcal{N}(n) = \mathcal{T}$ for every $n \in \mathbb{N}$. It suffices to show that every non-zero accessible subring D of P is in $l\mathcal{N}(n)$. By Proposition 2, there is a positive integer m such that $PD^m \subseteq D$. Since P is non-filial, there is $0 \neq a \in P$ such that $P \neq \langle 1 \rangle + Pa$. Let $0 \neq s \in D$. Then $0 \neq b = as^m \in D \cap Pa$. Since $P \neq \langle 1 \rangle + Pa$, $P \neq \langle 1 \rangle + Pb$. Hence, by Proposition 3, there is $c \in Pb$ with $acc_P(c) = \infty$. Obviously, $Pc \subseteq D$. By Proposition 2, $C = \langle c \rangle + Pc^n \in \mathcal{A}(n)$ and $C \subseteq D$. Hence, $0 \neq C \subseteq (l\mathcal{N}(n))(D)$. Every proper homomorphic image of D is in $\mathcal{P}$, so $D/(l\mathcal{N}(n))(D) \in \mathcal{P}$. Consequently, $D \in l\mathcal{N}(n)$. The result follows. $\square$

Theorem 3. Let P be a noetherian integrally closed domain, $a \in P$, $acc_P(a) = \infty$; $\mathcal{P}$ be the class of all proper homomorphic images of all non-zero accessible subrings of P and $\mathcal{S} = st(\mathcal{P} \cup \{R | R^2 = 0\})$. If $\mathcal{N} = \{\langle a^k \rangle + Pa^{k^2} | k = 1, 2, \ldots\} \cup \mathcal{S}$, then for every positive integer n, $l\mathcal{N} = ls\mathcal{N} = st\mathcal{N} \neq \mathcal{N}_n$.

Theorem 4. Let K be a field and $P = K[x]$, $\mathcal{P}$ be the class of all proper homomorphic images of all non-zero accessible subrings of P, $\mathcal{S} = st(\mathcal{P} \cup \{R | R^2 = 0\})$ and $\mathcal{N} = \{\langle f \rangle + Pf^{\deg\ f} | f \in P, \deg\ f \geq 2\} \cup \mathcal{S}$. Then $l\mathcal{N} = ls\mathcal{N} = st\mathcal{N} = \mathcal{N}_\omega = \mathcal{N}^\omega \neq \mathcal{N}_n = \mathcal{N}^n$ for every $n \in \mathbb{N}$ and the radical class $l\mathcal{N}$ is hereditary.

Theorem 5. Let $P_1, P_2, \ldots$ are pairwise non-isomorphic non-filial Dedekind rings, $\mathcal{K}$ is the class of all fields, $S = st(\mathcal{K} \cup \{R | R^2 = 0\})$, $\mathcal{A}(n)$ is the set of all precisely n-accessible subrings of P_n for $n \in \mathbb{N}$ and $\mathcal{N} = \mathcal{S} \cup \{\mathcal{A}(1)\} \cup \ldots$. Then $l\mathcal{N} = ls\mathcal{N} = st\mathcal{N} = \mathcal{N}_\omega = \mathcal{N}^\omega \neq \mathcal{N}_n = \mathcal{N}^n$ for every $n \in \mathbb{N}$ and the radical class $l\mathcal{N}$ is hereditary.

Question 1 (cf. [16]). Is for every class $\mathcal{N}$, $ls\mathcal{N} = \mathcal{N}^{\omega}$?

Question 2. Does for every natural number n there exist a radical class $\mathcal{N}$ with $ls\mathcal{N} = \mathcal{N}^{n+1} \neq \mathcal{N}^{n}$?

Question 3. Does there exist a radical class class $\mathcal{N}$ such that $ls\mathcal{N} = \mathcal{N}^{\omega} \neq \mathcal{N}^{n}$ for every natural number n?

Question 4. Does there exist a radical class class $\mathcal{N}$ such that $ls\mathcal{N} \neq \mathcal{N}^{\alpha}$ for every ordinal α?

The history of ADS problem:

- 1968, Heinicke: $P = F[x,t]$, $A_n = \langle x\rangle + Px^n$ for $n = 1, 2, \ldots$.
- 1982, Beidar: $P = \mathbb{Z}[i]$, p-prime of the form $4k+3$ and $A_n = \langle p\rangle + Pp^n$ for $n = 1, 2, \ldots$.
- 1984, Lvov and Sidorov: $A_n = F[a] + Pa^n$, $\dim_F(P/Pa) \geq 2$,
  P-completely normal F-algebra such that $\deg tr_F(P) = 1$.
- Watters: K is a proper field extension of a field F, $A = K[[x]]$, $A_n = xF[[x]] + Ax^n$.
- 1987, Guo Jin Yun: $F = \mathbb{Z}/(p)$, $P = F[x]$, $A_n = \langle x^n\rangle + Px^{n^2}$.
- 1988, Beidar: K-field, $P = K[x]$, $A_{n,f} = \langle x^r f\rangle + Pfx^{rn}$, $r \geq 2$, $f \in K[x]$, $f(0) \neq 0$.

Theorem ([1,17]). If $\mathcal{N}$ is a class of M-nilpotent rings then $l\mathcal{N} = ls\mathcal{N} = \mathcal{N}_3 = \mathcal{N}^3$.

Theorem ([16]). If $\mathcal{M}$ is hereditary radical then $ls\mathcal{M} = \mathcal{M}^2$.

Theorem ([16]). (i). If the class $\mathcal{N}$ is hereditary then $ls\mathcal{N} = \mathcal{N}^4$.
(ii). If the class $\mathcal{N}$ is hereditary and contains all trivial rings then $ls\mathcal{N} = \mathcal{N}^3$.

Theorem ([4,18]). (i). If the class $\mathcal{N}$ is hereditary then $l\mathcal{N} = \mathcal{N}^3$.
(ii). If the class $\mathcal{N}$ is hereditary and contains all trivial rings then $l\mathcal{N} = \mathcal{N}^2$.

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