Accessible subrings and Kurosh's chains of associative rings

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Abstract. Let $\mathcal{N}$ be a homomorphically closed class of associative rings. Put $\mathcal{N}_1 = \mathcal{N}^1 = \mathcal{N}$ and for ordinals $\alpha \geq 2$, define $\mathcal{N}_\alpha$, $(\mathcal{N}^\alpha)$ to be the class of all associative rings R such that every non-zero homomorphic image of R contains a non-zero ideal (left ideal) in $\mathcal{N}_\beta$ for some $\beta < \alpha$. In this way we obtain a chain $\{\mathcal{N}_\alpha\}$ $(\{\mathcal{N}^\alpha\})$, the union of which is equal to the lower radical class $l\mathcal{N}$ (lower left strong radical class $ls\mathcal{N}$) determined by $\mathcal{N}$. The chain $\{\mathcal{N}_\alpha\}$ is called *Kurosh's chain* of $\mathcal{N}$.
In [11] Suliński, Anderson and Divinsky studied the Kurosh's chain in the universal class of associative rings. They described the classes $\mathcal{N}_\alpha$ in terms of accessible subrings and proved that the chain stabilizes at the first limit ordinal ω. They asked whether, for each ordinal $\alpha \leq \omega$, there exists a class $\mathcal{N}$ such that $l\mathcal{N} = \mathcal{N}_\alpha \neq \mathcal{N}_\beta$ for $\beta < \alpha$. The question turned out to be very interesting and challenging. In [7], Heinicke answered it in the positive for $\alpha = \omega$. However, the problem for positive integers resisted effords of many authors for a long time. Various studies, except some general results, effected in solving the problem for small integers and determining when the chain stabilizes for some specific classes $\mathcal{N}$. The general problem was finally solved in the positive by Beidar in [3]. After that, several new examples were found (see [1], [4], [9], [1! 0], [12]). All of them were not easy to handle and required quite complicated arguments. Developing some ideas of the mentioned papers (mainly those of [10]), we construct new general examples. The generality allows us to avoid particular calculations which, we hope, makes the arguments clearer and more visible. It also allows us to construct radicals which satisfy some extra properties.

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